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Approximation by Periodic Spline Interpolants on Uniform Meshes[1]

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1. INTRODUCTION

If nothing is known about the function $x(t)$ but its values at a finite number of points and a bound for $\int_0^1 |x^{(r)}(t)|^2\,dt$ (for some positive r), then its $2r$-spline interpolant $Sx(t)$ is the best approximant (estimant). "Best" means that for any linear functional $u(x)$, for example $u(x) = x(\tau)$, the value $u(Sx)$ is the median of all values $u(x)$ consistent with the given data. The optimality of spline interpolation in this sense follows directly from the general theory of optimal approximation and estimation as established in [*1*, *2*]. Many other aspects of approximation by spline interpolants have been studied (for references see [*3*], [*7*] and [*8*]).

In this paper we consider periodic functions $x(t)$ and n interpolation points equally spaced in an interval of periodicity. Sx is said to be a $2r$-spline interpolant of x if Sx is periodic, has a continuous derivative of order $2r-2$, is an algebraic polynomial of degree $\leqslant 2r-1$ between knots t_k (the interpolation points), and $Sx(t_k) = x(t_k)$. The usual cubic splines appear as 4-splines in this notation. We establish explicit formulas for Sx and for $u(Sx)$, where the functional u represents interpolation, differentiation, quadrature, or a Fourier coefficient. No matrix inversion is needed to compute Sx or $u(Sx)$ if use is made of certain numerical coefficients (depending on r and n), whose explicit form is given [Sec. 2–4], and which can readily be computed. Especially noteworthy is the simple approximate value for the Fourier coefficient $\alpha_k = \int_0^1 x(t)\,e^{-2\pi ikt}\,dt$ of the function, determined from the spline interpolant:

$$\alpha_k \approx (\zeta_k/n) \sum_{\nu=0}^{n-1} x(\nu/n)\,e^{-2\pi i\nu k,n}, \quad \zeta_k^{-1} = \sum_{l=-\infty}^{\infty} (1 - ln/k)^{-2r}$$

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(Section 4). It differs from the simplest approximation only by the factor ζ_k. We also find optimal error bounds, asymptotic expressions for the error as the number of interpolation points becomes large, and convergence properties of the spline interpolants Sx and their derivatives [Sec. 6–11].

Basic for our analysis of approximation by periodic spline functions turn out to be the interpolants $b_\nu(t)$ of the functions $\exp(2\pi i\nu t)$ $(\nu = 0, 1, 2, \ldots, n-1)$ Section 5). The piecewise polynomial functions $b_\nu(t)$ with knots at m/n $(m = 0, \pm1, \pm2, \ldots)$ inherit many of the properties of the functions $\exp(2\pi i\nu t)$ that they interpolate. In particular,

$$b_\nu(t + 1/n) = e^{2\pi i\nu/n}\, b_\nu(t),$$

$|b_\nu(t)| \leqslant 1$, etc. Explicit formulas in terms of the Bernoulli functions $\mathring{B}_{2r}(t)$ (the periodic extension of the Bernoulli polynomial restricted to $0 \leqslant t \leqslant 1$) and the Fourier series for the $b_\nu(t)$ are given, and it is shown that they and their derivatives of order $\leqslant 2r - 1$ are orthogonal in the same sense as the functions $\exp(2\pi i\nu t)$ (see Section 5). If $x(t)$ has the absolutely convergent Fourier expansion $\sum \alpha_\nu \exp(2\pi i\nu t)$, then its $2r$-spline interpolant on a mesh of n equidistant points is $Sx(t) = S_r^n x(t) = \sum \alpha_\nu b_\nu(t)$ (Section 7). Making use of these representations, we find that the remainder $x(t) - S_r^n x(t)$ is, in the class of functions x restricted by $\sum |\nu|^p |\alpha_\nu| < \infty$ for some p, $0 \leqslant p \leqslant 2r$, of order $0(n^{-p})$ uniformly in t, and the sth derivative of this remainder is, for $0 \leqslant s \leqslant p$, of order $0(n^{-p+s})(o(1)$ if $s = p)$, (Theorem 7.1). If $p = 2r$, $s \leqslant 2r - 1$ and $x^{(s)}(t) - (S_r^n x)^{(s)}(t) = o(n^{-2r+s})$, then $x(t)$ is constant. As a by-product of this error analysis appears a formula for computing the derivative $x^{(2r)}$ as the limit of a remainder. Indeed

$$x^{(2r)}(0) = \theta_r \lim_{n\to\infty} n^{2r}[x(1/2n) - S_r^n x(1/2n)],$$

where θ_r is a simple numerical factor (Equation 7.23). The root mean-square error $\left\{\int_0^1 |x^{(s)}(t) - (S_r^n x)^{(s)}(t)|^2\, dt\right\}^{1/2}$ is, in the class of functions x restricted by $\sum |\nu|^{2p} |\alpha_\nu|^2 < \infty$ for some p, $\frac{1}{2} < p \leqslant 2r$, of order $0(n^{-p+s})$ for $s < p$ (Theorem 8.1). If $p = 2r$ and

$$\left\{\int_0^1 |x^{(s)}(t) - (S_r^n x)^{(s)}(t)|^2\, dt\right\}^{1/2} = o(n^{-2r+s}) \text{ for some } s,\ 0 \leqslant s \leqslant 2r - 1,$$

then $x(t)$ is constant. If $p = r$, that is, if we deal with the class of functions with an upper bound on $\int_0^1 |x^{(r)}(t)|^2\, dt$ given, then $S_r^n x(t)$ is the best estimation of $x(t)$ [see introductory remark], and

$$\int_0^1 |x^{(r)}(t) - (S_r^n x)^{(r)}(t)|^2\, dt = o(1) \quad \text{as } n \to \infty$$

(Theorem 8.2). From the order of convergence of the spline approximations $S_r^n x$ to x one can infer smoothness properties of the function. Thus, if

$\left\{\int_0^1 |x(t) - S_r^n x(t)|^2 dt\right\}^{1/2} = O(n^{-q})$ for some $q > 1$, then $\sum |\nu|^{2p} |\alpha_\nu|^2 < \infty$ for the largest integer p smaller than q (Theorem 8.3).

Uniform approximation in the class of functions x restricted by $\sum |\nu|^{2p} |\alpha_\nu|^2 < \infty$ is slightly less accurate than mean-square approximation. In this case,

$$|x^{(s)}(t) - (S_r^n x)^{(s)}(t)| = O(n^{-p+s+1/2}) \text{ for } s < p - \tfrac{1}{2}$$

(Theorem 9.1). That this is the precise order of error is also proved. This is done in connection with the problem to determine, for the functionals u mentioned above, the maximum deviation of $u(x)$ from its median value $u(S_r^n x)$ in the class of periodic functions x with $x(0), x(1/n), \ldots, x(1 - 1/n)$, and a bound on $\int_0^1 |x^{(r)}(t)|^2 dt$ given. For example, it is proved that $\lim n^{r-3/2} \sup |x'(0)|$, where the supremum is taken over the class of periodic functions x with $x(0) = x(1/n) = \ldots = x(1 - 1/n) = 0$ and $\int_0^1 |x^{(r)}(t)|^2 dt \leqslant 1$, exists and is positive, and its value is determined (Theorem 11.2). Similar results are derived for the interpolation, quadrature, and Fourier coefficient functionals (Section 11).

2. The Cardinal Interpolants

Let $\xi_0, \xi_1, \ldots, \xi_{n-1}$ be $n \geqslant 1$ given (real or complex) numbers. We wish to construct the $2r$-spline (r a fixed positive integer) $s(t) = s_r^n(t) = s_r^n(t;\xi)$ of period 1 with knots [discontinuities of the $(2r-1)$st derivative] at the points $0, \pm 1/n, \pm 2/n, \ldots$, which takes on the value ξ_ν at the point ν/n, $\nu = 0, 1, \ldots, n-1$. Thus we require

$$\begin{aligned}
&\text{(i)}\ s \in \mathscr{C}_{2r-2}\\
&\text{(ii)}\ s(t+1) = s(t), \quad -\infty < t < \infty\\
&\text{(iii)}\ s^{(2r)}(t) = 0, \quad t \neq 0, \pm 1/n, \pm 2/n, \ldots\\
&\text{(iv)}\ s(\nu/n) = \xi_\nu, \quad \nu = 0, 1, \ldots, n-1.
\end{aligned} \tag{2.1}$$

The existence and uniqueness of the function s satisfying conditions (2.1) follows from the fact that the problem of minimizing the integral

$$\int_0^1 |x^{(r)}(t)|^2 dt \tag{2.2}$$

among the functions $x \in \mathscr{C}_{r-1}$ of period 1 for which $x(\nu/n) = \xi_\nu$, $\nu = 0, 1, \ldots, n-1$, has exactly one solution, $x = s$ (see [*1*]).

We expand $s(t)$ first with respect to the basis formed by the functions

$$1, \mathring{B}_{2r}(t - \nu/n) \qquad \nu = 0, 1, \ldots, n-1. \tag{2.3}$$

Here $\mathring{B}_{2r}(t)$ is the Bernoulli function of period 1 which is the periodic extension of the Bernoulli polynomial $B_{2r}(t)$ restricted to the interval $0 \leqslant t \leqslant 1$. Thus, (see [4])

$$\begin{aligned} \mathring{B}_{2r}(t) &= B_{2r}(t) = \sum_{p=0}^{2r} \binom{2r}{p} B_p t^{2r-p} && 0 \leqslant t \leqslant 1 \\ \mathring{B}_{2r}(t+1) &= \mathring{B}_{2r}(t) && -\infty < t < \infty, \end{aligned} \tag{2.4}$$

where B_p is the pth Bernoulli number, $B_p = B_p(0)$ [in particular, $B_{2p+1} = 0$ for $p = 1,2,\ldots$]. Since $B_p(t) = (-1)^p B_p(1-t)$ and $B'_{p+1}(t) = (p+1)B_p(t)$ ($p = 0, 1,2,\ldots$), it follows that $\mathring{B}_{2r}$ is an even function in $\mathscr{C}_{2r-2}$, $\mathring{B}_{2r}^{(2r+1)}(t) = 0$ for $t \neq 0, \pm 1, \pm 2, \ldots$, and

$$\mathring{B}_{2r}^{(2r-1)}(0+) - \mathring{B}_{2r}^{(2r-1)}(0-) = -(2r)!. \tag{2.5}$$

We also mention the useful identity (see [4])

$$\mathring{B}_{2r}(nt) = n^{2r-1} \sum_{\nu=0}^{n-1} \mathring{B}_{2r}(t - \nu/n). \tag{2.6}$$

These properties of $\mathring{B}_{2r}$ are evident from the Fourier expansion (see [4]), which might serve for the definition of $\mathring{B}_{2r}$:

$$\begin{aligned} \mathring{B}_{2r}(t) &= \frac{(-1)^{r-1}(2r)!}{(2\pi)^{2r}} 2 \sum_{k=1}^{\infty} \frac{\cos 2\pi kt}{k^{2r}} \\ &= \frac{(-1)^{r-1}(2r)!}{(2\pi)^{2r}} \sum_k{}' \frac{e^{2\pi ikt}}{k^{2r}}. \end{aligned} \tag{2.7}$$

Here and in the following $\sum_k{}'$ stands for $\lim_{l\to\infty}(\sum_{k=1,\ldots,l} + \sum_{k=-1,\ldots,-l})$.

The following expression of $B_{2r}(t)$ in powers of $t(1-t)$ is well suited for computation (see [4])

$$B_{2r}(t) = (-1)^r \sum_{p=0}^{r} B_{r,p}[t(1-t)]^p. \tag{2.8}$$

The coefficients involved are obtained recursively from

$$\begin{aligned} B_{r,0} &= (-1)^r B_{2r} \\ p(p+1) B_{r,p+1} &= 2p(2p-1) B_{r,p} - 2p(2p-1) B_{r-1,p-1}. \end{aligned} \tag{2.9}$$

Particular values are $B_{1,1} = 1$, $B_{r,1} = 0$ for $r > 1$, $B_{r,r} = 1$.

To obtain the spline function $s(t)$ defined by conditions (2.1) we set

$$s(t) = \eta + \sum_{\nu=0}^{n-1} \eta_\nu \mathring{B}_{2r}(t - \nu/n) \tag{2.10}$$

where the coefficients $\eta, \eta_0, \ldots, \eta_{n-1}$ are determined so that

$$\sum_{\nu=0}^{n-1} \eta_\nu = 0 \tag{2.11}$$

and

$$s(\nu/n) = \xi_\nu \qquad \nu = 0, 1, \ldots, n-1. \tag{2.12}$$

Condition (2.11) implies $s^{(2r)}(t) = (2r)! \sum \eta_\nu = 0$ for $t \neq 0, \pm 1, \pm 2, \ldots$. Thus if (2.11) and (2.12) are satisfied, then s is the desired spline interpolant.

If we substitute $t = \mu/n$ in (2.10), sum over $\mu = 0, 1, \ldots, n-1$ and use (2.6) [with $t = 0$], we obtain on account of (2.11) and (2.12)

$$\sum_{\mu=0}^{n-1} \xi_\mu = n\eta + n^{1-2r} B_{2r} \sum_{\nu=0}^{n-1} \eta_\nu = n\eta;$$

thus

$$\eta = (1/n) \sum_{\mu=0}^{n-1} \xi_\mu. \tag{2.13}$$

The interpolation conditions (2.12) now give

$$\sum_{\mu=0}^{n-1} \sigma_{\nu-\mu} \eta_\mu = \xi_\nu - \eta \qquad \nu = 0, 1, \ldots, n-1 \tag{2.14}$$

where we have set $\sigma_m = \sigma^n_{r,m}$:

$$\sigma_m = \mathring{B}_{2r}(m/n) \qquad m = 0, \pm 1, \pm 2, \ldots. \tag{2.15}$$

The matrix of the linear system (2.14) is a circulant, its n^2 elements are replica of $\sigma_0, \sigma_1, \ldots, \sigma_{n/2}$ since $\sigma_m = \sigma_{-m}$, $\sigma_{m+n} = \sigma_m$ $(m = 0, 1, 2, \ldots)$. These numbers can be calculated by the use of (2.8),

$$\sigma^n_{r,m} = (-1)^r \sum_{p=0}^{r} B_{r,p} m^p (n-m)^p / n^{2p}. \tag{2.16}$$

The calculations can be reduced by making use of the obvious relation

$$\sigma^{kn}_{r,k\nu} = \sigma^n_{r,\nu} \qquad k = 1, 2, \ldots.$$

The inverse of the matrix $\{\sigma_{\nu-\mu}\}$ is also a circulant, which we denote as $\{\rho_{\nu-\mu}\}$. Again we have $\rho_m = \rho_{-m}$, $\rho_{m+n} = \rho_m$, so that the n^2 elements of $\{\rho_{\nu-\mu}\}$ are replica of $\rho_0, \rho_1, \ldots, \rho_{n/2}$. To calculate these numbers, we first observe that the n-vectors

$$\{1, \epsilon_n^{\nu}, \epsilon_n^{2\nu}, \ldots, \epsilon_n^{(n-1)\nu}\},\ \epsilon_n = e^{2\pi i/n} \qquad \nu = 0, 1, \ldots, n-1 \tag{2.17}$$

are eigenvectors of the matrix $\{\sigma_{\nu-\mu}\}$, and the corresponding eigenvalues are (for simplicity we assume n is even)

$$\begin{aligned}\lambda_\nu &= \sum_{m=0}^{n-1} \sigma_m \epsilon_n^{m\nu} \\ &= \sigma_0 + 2[\sigma_1 \cos 2\pi\nu/n + \sigma_2 \cos 4\pi\nu/n + \ldots \\ &\qquad + \sigma_{n/2-1} \cos 2\pi(n/2-1)\nu/n] + (-1)^\nu \sigma_{n/2}.\end{aligned} \tag{2.18}$$

Clearly $\lambda_\nu = \lambda_{n-\nu}$. Using Fourier series (2.7), we find the following expression for λ_ν:

$$\lambda_\nu = (-1)^{r-1}(2r)!\, n^{-2r+1}(2\pi)^{-2r} \sum_{k=-\infty}^{\infty} (k - \nu/n)^{-2r} \tag{2.19}$$

(for $\nu = 0$ the term with $k = 0$ is to be omitted in the sum). Observing that the vectors (2.17) satisfy the orthogonality relations

$$\sum_{m=0}^{n-1} \epsilon_n^{m\mu}\, \epsilon_n^{-m\nu} = n\delta_{\mu,\nu},$$

we find for the ρ_ν the explicit expression

$$\begin{aligned} n\rho_\nu &= \sum_{m=0}^{n-1} \lambda_m^{-1}\, \epsilon_n^{m\nu} \\ &= \lambda_0^{-1} + 2[\lambda_1^{-1} \cos 2\pi\nu/n + \lambda_2^{-1} \cos 4\pi\nu/n + \dots \\ &\qquad + \lambda_{n/2-1}^{-1} \cos 2\pi(n/2-1)\,\nu/n] + (-1)^\nu \lambda_{n/2}^{-1}. \end{aligned} \tag{2.20}$$

As with the σ's the calculation of the ρ's is simplified by making use of the relations

$$\lambda_{r,k\nu}^{kn} = k^{-2r+1}\lambda_{r,\nu}^{n},\ \rho_{r,k\nu}^{kn} = k^{-2r}\rho_{r,\nu}^{n} \qquad k = 1, 2, \dots . \tag{2.20a}$$

With the numbers ρ found, we have the explicit inversion of system (2.14)

$$\eta_\nu = \sum_{\mu=0}^{n-1} \rho_{\nu-\mu}(\xi_\mu - \eta) \qquad \nu = 0, 1, \dots, n-1.$$

Since by (2.6), (2.15), (2.18), (2.20)

$$\begin{aligned} \sum_{\nu=0}^{n-1} \rho_\nu &= 1 \Big/ \sum_{\nu=0}^{n-1} \sigma_\nu \\ &= \lambda_0^{-1} = n^{2r-1} B_{2r}^{-1}, \end{aligned} \tag{2.21}$$

we have more explicitly

$$\eta_\nu = \sum_{\mu=0}^{n-1} \rho_{\nu-\mu}\,\xi_\mu - n^{2r-1}\eta/B_{2r},\ \eta = (1/n)\sum_{\mu=0}^{n-1} \xi_\mu . \tag{2.22}$$

This completes the calculation of the interpolating spline s.

If we let $s_\nu = s_{r,\nu}^n$ $(\nu = 0, 1, \dots, n-1)$ be the *cardinal interpolating spline* satisfying

$$s_\nu(\mu/n) = \delta_{\mu,\nu} \qquad \mu, \nu = 0, 1, \dots, n-1 \tag{2.23}$$

in place of (2.1(iv)), then by (2.22) the corresponding coefficients are $\eta_\mu = \rho_{\nu-\mu} - n^{2r-1}\eta/B_{2r}$, $\eta = n^{-1}$; hence

$$\begin{aligned} s_\nu(t) &= 1/n + \sum_{\mu=0}^{n-1} (\rho_{\nu-\mu} - n^{2r-2}/B_{2r})\,\mathring{B}_{2r}(t - \mu/n) \\ &= (1/n)\,[1 - \mathring{B}_{2r}(nt)/B_{2r}] + \sum_{\mu=0}^{n-1} \rho_{\nu-\mu}\,\mathring{B}_{2r}(t - \mu/n). \end{aligned} \tag{2.24}$$

As one would expect, the s_ν can be expressed as translates of the one even function s_0:

$$s_\nu(t) = s_0(t - \nu/n) \qquad \nu = 0, 1, \ldots, n-1$$

$$s_0(t) = 1/n + \sum_{\nu=0}^{n-1} (\rho_\nu - n^{2r-2}/B_{2r})\, \mathring{B}_{2r}(t + \nu/n)$$

$$= (1/n)\,[1 - \mathring{B}_{2r}(nt)/B_{2r}] + \sum_{\nu=0}^{n-1} \rho_\nu \mathring{B}_{2r}(t + \nu/n). \tag{2.25}$$

3. Interpolation, Differentiation, Quadrature

a. If $x(t)$ is the function to be interpolated, with $\xi_\nu = x(\nu/n)$ given ($\nu = 0, 1, \ldots, n-1$), the spline interpolation of $x(t)$ at $t = \tau$ is denoted by $Sx(\tau)$, and is given by

$$Sx(\tau) = \sum_{\nu=0}^{n-1} x(\nu/n)\, s_0(\tau - \nu/n) \tag{3.1}$$

where s_0 is given in (2.25). $S = S_r^n$ is to be considered a linear operator, transforming general periodic functions into periodic $2r$-splines.

b. The spline derivative of $x(t)$ at $t = \tau$ is given by

$$DSx(\tau) = (Sx)'(\tau) = \sum_{\nu=0}^{n-1} x(\nu/n)\, s_0'(\tau - \nu/n), \tag{3.2}$$

where s_0' is obtained from (2.25):

$$s_0'(t) = 2r \sum_{\nu=0}^{n-1} (\rho_\nu - n^{2r-2}/B_{2r})\, \mathring{B}_{2r-1}(t + \nu/n)$$

$$= 2r\left[-\mathring{B}_{2r-1}(nt)/B_{2r} + \sum_{\nu=0}^{n-1} \rho_\nu \mathring{B}_{2r-1}(t + \nu/n)\right]. \tag{3.3}$$

If τ is one of the interpolation points, say $\tau = 0$, then (3.2) gives the following approximation to $x'(0)$:

$$(Sx)'(0) = \sum_{\nu=0}^{n-1} \delta_\nu x(\nu/n), \quad \delta_\nu = 2r \sum_{\mu=0}^{n-1} \rho_{\nu+\mu} B_{2r-1}(\mu/n). \tag{3.4}$$

c. The spline quadrature value of $\int_{-T}^{T} x(t)\,dt$ is given by

$$\int_{-T}^{T} Sx(t)\,dt = \sum_{\nu=0}^{n-1} x(\nu/n) \int_{-T}^{T} s_0(t - \nu/n)\,dt \tag{3.5}$$

where

$$\int_{-T}^{T} s_0(t - \nu/n)\,dt = 2\tau/n + (2r + 1)^{-1} \sum_{\mu=0}^{n-1} (\rho_{\nu-\mu} - n^{2r-2}$$

$$/B_{2r})\,[\mathring{B}_{2r+1}(\tau + \mu/n) + \mathring{B}_{2r-1}(\tau - \mu/n)]. \tag{3.6}$$

For the special case $\tau = 1/n$ we obtain the quadrature formula

$$\int_{-1/n}^{1/n} Sx(t)\,dt = \sum_{\nu=0}^{n-1} \kappa_\nu x(\nu/n),$$

$$\kappa_\nu = 2n^{-2} + (2r+1)^{-1} \sum_{\mu=0}^{n-1} (\rho_{\nu+\mu-1} - \rho_{\nu+\mu+1})\, B_{2r+1}(\mu/n). \tag{3.7}$$

4. Fourier Coefficients

The spline approximation of the Fourier coefficient $\int_0^1 x(t)\exp(-2\pi ikt)\,dt$ $(k = 0, \pm 1, \pm 2, \ldots)$ is

$$\int_0^1 Sx(t)\,e^{-2\pi ikt}\,dt = \sum_{\nu=0}^{n-1} x(\nu/n) \int_0^1 s_0(t - \nu/n)\,e^{-2\pi ikt}\,dt$$

$$= \sum_{\nu=0}^{n-1} \epsilon_n^{-k\nu} x(\nu/n) \int_0^1 s_0(t)\,e^{-2\pi ikt}\,dt. \tag{4.1}$$

We put

$$\int_0^1 s_0(t)\,e^{-2\pi ikt}\,dt = \int_0^1 s_0(t)\,e^{2\pi ikt}\,dt = \int_0^1 s_0(t)\cos 2\pi kt\,dt \tag{4.2}$$

$$= \hat{s}_0(k) \qquad k = 0, 1, 2, \ldots$$

and proceed to determine these coefficients. By (2.7), for $k \neq 0$

$$\int_0^1 \mathring{B}_{2r}(t + \nu/n)\,e^{-2\pi ikt}\,dt = \epsilon_n^{k\nu} \int_0^1 B_{2r}(t)\,e^{-2\pi ikt}\,dt$$

$$= (-1)^{r-1}(2r)!\,(2\pi k)^{-2r}\,\epsilon_n^{k\nu};$$

hence by (2.25)

$$\hat{s}_0(k) = (-1)^{r-1}(2r)!\,(2\pi k)^{-2r} \sum_{\nu=0}^{n-1} (\rho_\nu - n^{2r-2}/B_{2r})\,\epsilon_n^{k\nu}. \tag{4.3}$$

By the definition of ρ_ν, λ_ν and ϵ_n we have

$$\sum_{\nu=0}^{n-1} \rho_\nu\,\epsilon_n^{k\nu} = \lambda_k^{-1} \qquad k = 0, 1, 2, \ldots \tag{4.4}$$

$$\sum_{\nu=0}^{n-1} \epsilon_n^{k\nu} = n \qquad \text{if } k \equiv 0 \pmod{n}$$

$$= 0 \qquad \text{if } k \not\equiv 0 \pmod{n} \tag{4.5}$$

where we have set $\lambda_{k+n} = \lambda_n$ $(k = 0, 1, 2, \ldots)$. Since $n^{2r-1}/B_{2r} = \lambda_0^{-1}$ [see (2.21)], (4.2), (4.3) and (4.4) give

$$\hat{s}_0(k) = (-1)^{r-1}(2r)!\,(2\pi k)^{-2r}\lambda_k^{-1} \qquad k \not\equiv 0 \pmod{n}$$

$$= 0 \qquad k \equiv 0 \pmod{n},\ k \neq 0 \tag{4.6}$$

$$= n^{-1} \qquad k = 0.$$

These are the Fourier coefficients of s_0.

If (4.6) is used in (4.1), one obtains the following explicit formulas for the spline approximation of the Fourier coefficients of the function $x(t)$:

$$\begin{aligned}\int_0^1 Sx(t)\,e^{-2\pi ikt}\,dt &= (1/n)\sum_{\nu=0}^{n-1} x(\nu/n) & k&=0\\ &=0 & k&\equiv 0\,(\mathrm{mod}\,n),\ k\neq 0 \qquad (4.7)\\ &=(-1)^{r-1}(2r)!\,(2\pi k)^{-2r}\lambda_k^{-1}\sum_{\nu=0}^{n-1} x(\nu/n)\,\epsilon_n^{-k\nu}, & k&\not\equiv 0\,(\mathrm{mod}\,n).\end{aligned}$$

If we use the expression (2.19) for λ_k in (4.7), we obtain the following simple formula for the Fourier coefficients:

$$\int_0^1 Sx(t)\,e^{-2\pi ikt}\,dt = (1/n)\sum_{\nu=0}^{n-1} x(\nu/n)\,\epsilon_n^{-k\nu}\Big/\sum_{l=-\infty}^{\infty}(1-ln/k)^{-2r}, \quad k\not\equiv 0\,(\mathrm{mod}\,n). \tag{4.8}$$

It is interesting to observe that the commonly used approximation

$$(1/n)\sum_{\nu=0}^{n-1} x(\nu/n)\,\epsilon_n^{-k\nu}$$

(which results from the trapezoidal rule) turns out to be a biased estimate in the class of functions x with a known bound on $\int_0^1 |x^{(r)}(t)|^2\,dt$, the bias factor $\sum_l (1-ln/k)^{-2r}$ being the larger, if $|k|\leqslant n/2$, the smaller r is. From (4.7) it also follows that if $k_1\equiv k_2\not\equiv 0\,(\mathrm{mod}\,n)$, then

$$\int_0^1 Sx(t)\,e^{-2\pi ik_1t}\,dt \div \int_0^1 Sx(t)\,e^{-2\pi ik_2t}\,dt = k_2^{2r}\div k_1^{2r}. \tag{4.9}$$

The trapezoidal rule gives the same value for the k_1th and k_2th Fourier coefficients, which is clearly useless. The rate of decrease expressed in (4.9) is the expected one for the class of functions x with a bound on $\int_0^1 |x^{(r)}(t)|^2\,dt$. In [*10*], Collatz and Quade obtain the same result for the Fourier coefficients, but with a different expression for the bias factor.

5. The Exponential Interpolants

We now introduce the important functions $b_\nu = b_{r,\nu}^n$ $(\nu = 0,\pm 1,\pm 2,\ldots)$ defined as

$$\begin{aligned} b_\nu(t) &= 1 & \nu&\equiv 0\,(\mathrm{mod}\,n)\\ b_\nu(t) &= \lambda_\nu^{-1}\sum_{m=0}^{n-1}\epsilon_n^{\nu m}\hat{B}_{2r}(t-m/n) & & \qquad (5.1)\\ &=\sum_{m=0}^{n-1}\epsilon_n^{\nu m}\hat{B}_{2r}(t-m/n)\Big/\sum_{m=0}^{n-1}\epsilon_n^{\nu m}B_{2r}(m/n) & \nu&\not\equiv 0\,(\mathrm{mod}\,n).\end{aligned}$$

Clearly, $b_{\nu+n} = b_\nu$ and $b_{-\nu} = \bar{b}_\nu$. The b_ν are $2r$-splines since

$$\sum_{m=0}^{n-1} \epsilon_n^{\nu m} = 0 \quad \text{if } \nu \not\equiv 0 \,(\text{mod}\, n).$$

They have the fundamental property

$$b_\nu(t + 1/n) = \epsilon_n{}^\nu b_\nu(t) = e^{2\pi i\nu/n} b_\nu(t) \qquad \nu = 0, \pm 1, \pm 2, \ldots. \tag{5.2}$$

Since $b_\nu(0) = 1$, it follows from (5.2) that

$$b_\nu(m/n) = \epsilon_n^{\nu m} = e^{2\pi i\nu m/n} \qquad \nu = 0, 1, \ldots, n-1. \tag{5.3}$$

Thus $b_\nu(t)$ is the $2r$-spline interpolant of the function $\exp(2\pi i\nu t)$ [and also of $\exp[2\pi i(\nu + kn)t]$, $k = 0, \pm 1, \pm 2, \ldots$], and $\text{Re}\, b_\nu(t)$, $\text{Im}\, b_\nu(t)$ interpolate $\cos 2\pi\nu t$, $\sin 2\pi\nu t$, respectively. Therefore, also,

$$b_\nu(t) = \sum_{m=0}^{n-1} \epsilon_n^{\nu m} s_0(t - m/n) \qquad \nu = 0, \pm 1, \pm 2, \ldots. \tag{5.4}$$

Conversely, s_0 may be expressed in terms of $b_0, \ldots, b_{n-1}$. By (5.4)

$$s_0(t) = (1/n) \sum_{\nu=0}^{n-1} b_\nu(t). \tag{5.5}$$

Hence the spline interpolant Sx may be expressed in terms of the b_ν. By (5.2) and (5.5)

$$s_0(t - m/n) = (1/n) \sum_{\nu=0}^{n-1} \epsilon_n^{-\nu m} b_\nu(t)$$

and this together with (3.1) gives

$$Sx(t) = \sum_{\nu=0}^{n-1} \hat{\xi}_\nu b_\nu(t)$$

$$\hat{\xi}_\nu = (1/n) \sum_{\mu=0}^{n-1} \epsilon_n^{-\mu\nu} \xi_\mu = (1/n) \sum_{\mu=0}^{n-1} \epsilon_n^{-\mu\nu} x(\mu/n). \tag{5.6}$$

Formula (5.6) shows that $x(t)$ has the same spline interpolant as the trigonometric polynomial

$$\sum_{\nu=0}^{n-1} \hat{\xi}_\nu e^{2\pi i\nu t}, \quad \hat{\xi}_\nu = (1/n) \sum_{\mu=0}^{n-1} \epsilon_n^{-\mu\nu} x(\mu/n) \tag{5.7}$$

[independent of r]. (5.7) is clearly an interpolating polynomial of $x(t)$.

The Fourier expansion of b_ν is easily obtained from (2.7), using (2.19):

$$\begin{aligned} b_\nu(t) &= \frac{(-1)^{r-1}(2r)!}{(2\pi)^{2r}\lambda_\nu} \sum_k{}' \left[\sum_{m=0}^{n-1} \epsilon_n^{(\nu-k)m}\right] \frac{e^{2\pi ikt}}{k^{2r}} \\ &= \frac{(-1)^{r-1}(2r)!}{(2\pi)^{2r}\lambda_\nu} \sum_k \frac{e^{2\pi i(\nu-kn)t}}{(\nu - kn)^{2r}} \\ &= \sum_k (k - \nu/n)^{-2r} e^{2\pi i(\nu-kn)t} \Big/ \sum_k (k - \nu/n)^{-2r}, \quad \nu \not\equiv 0 \,(\text{mod}\, n). \end{aligned} \tag{5.8}$$

We also record the Fourier expansion of the derivatives $b_\nu^{(s)}$, $s = 1, 2, \ldots, 2r-1$:

$$b_\nu^{(s)}(t) = (-2\pi i n)^s \sum_k (k - \nu/n)^{-2r+s} e^{2\pi i(\nu - kn)t} \Big/ \sum_k (k - \nu/n)^{-2r}$$

$$\nu \not\equiv 0 \pmod n; \; s = 0, 1, \ldots, 2r-1. \tag{5.9}$$

The spline functions $b_\nu(t)$ ($\nu = 0, 1, \ldots, n-1$) and their derivatives $b_\nu^{(s)}(t)$ are orthogonal just like the functions $\exp(2\pi i \nu t)$ which they interpolate.[3] Indeed, by (5.9)

$$\int_0^1 b_\mu^{(s)}(t) \overline{b_\nu^{(s)}(t)}\, dt = 0 \quad \text{if } \mu \not\equiv \nu \pmod n, \; s = 0, 1, \ldots, 2r-1. \tag{5.10}$$

For the normalization factor we have by (5.9) and (2.19)

$$\int_0^1 |b_\nu^{(s)}(t)|^2\, dt = (2\pi\nu)^{2s} \left(1 + \sum_k{}' (1 - kn/\nu)^{-4r+2s}\right) \Big/ \left(1 + \sum_k{}' (1 - kn/\nu)^{-2r}\right)^2$$

$$\nu \not\equiv 0 \pmod n; \; s = 0, 1, \ldots, 2r-1. \tag{5.11}$$

For $s = r$, (5.11) reduces to

$$\int_0^1 |b_\nu^{(r)}(t)|^2\, dt = (2\pi\nu)^{2r} \Big/ \sum_k (1 - kn/\nu)^{-2r} \qquad \nu \not\equiv 0 \pmod n. \tag{5.12}$$

Since it is known that, among all the functions in the class $\mathscr{W}_r$ (periodic functions with square-integrable rth derivatives, see Section 6) which interpolate a function x_0, the $2n$-spline interpolant Sx_0 attains the minimal value of $\int_0^1 |x^{(r)}(t)|^2\, dt$, we conclude:

For no function x in $\mathscr{W}_r$ for which $x(k/n) = e^{2\pi i \nu k/n}$ $(k = 0, \pm 1, \pm 2, \ldots)$ is the value of $\int_0^1 |x^{(r)}(t)|^2\, dt$ smaller than the number (5.12), *and only for $x = b_\nu$ is this value attained.*

By (5.9), we have for the values of the derivatives at the knots

$$b_\nu^{(s)}(m/n) = \beta_\nu^{(s)} (2\pi i \nu)^s e^{2\pi i \nu m/n}$$

$$\beta_\nu^{(s)} = \beta_{r,\nu}^{(s)} = \left(1 + \sum_k{}' (1 - kn/\nu)^{-2r+s}\right) \left(1 + \sum_k{}' (1 - kn/\nu)^{-2r}\right)^{-1} \tag{5.13}$$

$$\nu \not\equiv 0 \pmod n; \; s = 0, 1, \ldots, 2r-2.$$

Thus, $b_\nu^{(s)}(t)$ interpolates the sth derivative of $\beta_\nu^{(s)} \exp(2\pi i \nu t)$ at the knots m/n, and $\operatorname{Re} b_\nu^{(s)}(t)$, $\operatorname{Im} b_\nu^{(s)}(t)$ interpolate the sth derivatives of $\beta_\nu^{(s)} \cos 2\pi\nu t$,

[3] The orthogonality property of periodic splines considered in [5] concerns splines on imbedded meshes, while (5.10) expresses orthogonality of splines interpolating orthogonal functions on the same mesh.

$\beta_\nu^{(s)} \sin 2\pi\nu t$. Since $b_\nu^{(2s)}$ is a $2(r-s)$-spline, and since the interpolating spline is unique, we conclude

$$b_{r,\nu}^{(2s)}(t) = \beta_{r,\nu}^{(2s)} b_{r-s,\nu}(t), \quad \nu \not\equiv 0 \,(\mathrm{mod}\, n);\ s = 1, 2, \ldots, r-1.$$

We have this relation for the derivatives of even order only because we have restricted ourselves only to splines of even order.

To calculate the piecewise constant $b_\nu^{(2r-1)}(t)$, we use (5.9) halfway between consecutive knots. We obtain

$$b_\nu^{(2r-1)}\overline{(m+\tfrac{1}{2}/n)} = \beta_\nu^{(2r-1)}(2\pi i\nu)^{2r-1}\, e^{2\pi i\nu(m+1/2)/n}$$

$$\beta_\nu^{(2r-1)} = \beta_{r,\nu}^{(2r-1)} = \left(1 + \sum_k{}' (-1)^k (1 - kn/\nu)^{-1}\right)\left(1 + \sum_k{}' (1 - kn/\nu)^{-2r}\right)^{-1} \quad (5.14)$$

$$\nu \not\equiv 0 \,(\mathrm{mod}\, n).$$

Thus, $b_\nu^{(2r-1)}(t)$ interpolates the $(2r-1)$th derivative of $\beta_\nu^{(2r-1)}\exp(2\pi i\nu t)$ at the points $t = (m+\frac{1}{2})/n$. The piecewise constant $b_\nu^{(2r-1)}(t)$ may be used to compute $b_\nu(t)$.

Because of the periodicity property (5.2), $b_\nu(t)$ need be computed only for $0 < t < 1/n$. Actually, the interval $0 < t < 1/2n$ is sufficient since we also have the symmetry property

$$b_\nu(1/2n + t) = \epsilon_n{}^\nu\, \overline{b_\nu(1/2n - t)}, \quad (5.15)$$

which follows directly from (5.1).

6. Bounds and Approximation Errors of the b_ν

From the Fourier expansion (5.8) one obtains immediately

Lemma 6.1

$$|b_\nu(t)| \leqslant 1, \qquad -\infty < t < \infty;\ \nu = 0, \pm 1, \pm 2, \ldots. \quad (6.1)$$

One also sees that if $\nu \not\equiv 0$ (mod n), then $|b_\nu(t)| = 1$ if and only if $t = m/n$ ($m = 0, \pm 1, \pm 2, \ldots$), that is, at the knots of b_ν. For the derivatives $b_\nu^{(s)}$ we do not have the least upper bounds; however by (5.9)

$$|b_\nu^{(s)}(t)| \leqslant \beta_{\nu *}^{(s)}(2\pi\nu)^s$$

$$\beta_{\nu *}^{(s)} = \left(1 + \sum_k{}' |1 - kn/\nu|^{-2r+s}\right)\Big/\left(1 + \sum_k{}' |1 - kn/\nu|^{-2r}\right) \quad (6.2)$$

$$\nu \not\equiv 0 \,(\mathrm{mod}\, n);\ s = 0, 1, \ldots, 2r-2.$$

We write $\beta_{\nu *}^{(s)}$ as a fraction whose denominator is

$$1 + \sum_{k=1}^{\infty} (1 + kn/\nu)^{-2r} + (n/\nu - 1)^{-2r} + \sum_{k=2}^{\infty} (kn/\nu - 1)^{-2r}$$

and whose numerator consists of the same terms, with the exponent $-2r$ replaced by $-2r+s$. To estimate $\beta_{\nu*}^{(s)}$ for $1 \leqslant \nu \leqslant n-1$ we use the inequalities

$$\sum_{k=1}^{\infty} (1+kn/\nu)^{-2r+s} < \int_0^{\infty} (1+xn/\nu)^{-2r+s}\,dx = (\nu/n)(2r-s-1)^{-1}$$

$$\sum_{k=2}^{\infty} (kn/\nu-1)^{-2r+s} < \int_1^{\infty} (xn/\nu-1)^{-2r+s}\,dx = (\nu/n)^{2r-s}(1-\nu/n)^{-2r+s+1}(2r-s-1)^{-1}. \tag{6.3}$$

Then

$$\beta_{\nu*}^{(s)} < \frac{1+\left(\frac{\nu}{n}\right)(2r-s-1)^{-1}+\left(\frac{\nu}{n}\right)^{2r-s}\left(1-\frac{\nu}{n}\right)^{-2r+s}+\left(\frac{\nu}{n}\right)^{2r-s}\left(1-\frac{\nu}{n}\right)^{-2r+s+1}(2r-s-1)^{-1}}{1+\left(\frac{\nu}{n}\right)^{2r}\left(1-\frac{\nu}{n}\right)^{-2r}}.$$

If $2\nu \leqslant n$, then since $(\nu/n)^{2r-s}(1-\nu/n)^{-2r+s} \leqslant 1$,

$$\begin{aligned}\beta_{\nu*}^{(s)} &< 1+(\nu/n)(2r-s-1)^{-1}+1+(1-\nu/n)(2r-s-1)^{-1}\\ &= 2+(2r-s-1)^{-1}\\ &< 3.\end{aligned}$$

If $2\nu > n$, then $(\nu/n)^{2r-s}(1-\nu/n)^{-2r+s} \leqslant (\nu/n)^{2r}(1-\nu/n)^{-2r}$, while $2r-s-1+\nu/n > 2r-s-\nu/n$. Making use of the inequality $(A_1+B_1)/(A_2+B_2) \leqslant B_1/B_2$ if $0 < A_1 \leqslant A_2$, $0 < B_2 \leqslant B_1$, (6.3) gives

$$\begin{aligned}\beta_{\nu*}^{(s)} &\leqslant (2r-s-1+\nu/n)(2r-s-1)^{-1}\\ &= 1+(\nu/n)(2r-s-1)^{-1}\\ &< 2.\end{aligned}$$

Thus, we have shown

$$\beta_{\nu*}^{(s)} \leqslant 3, \qquad \nu = 1,\ldots,n-1;\ s = 0,1,\ldots,2r-2. \tag{6.4}$$

To estimate $b_\nu^{(2r-1)}(t)$, we use (5.14):

$$|b_\nu^{(2r-1)}(t)| \leqslant \beta_{\nu*}^{(2r-1)}(2\pi\nu)^{2r-1}$$

$$\beta_{\nu*}^{(2r-1)} = |1+\sum_k{}' (-1)^k(1-kn/\nu)^{-1}|/|1+\sum_k{}' (1-kn/\nu)^{-2r}| \tag{6.5}$$

Then, for $1 \leqslant \nu \leqslant n-1$,

$$\beta_{\nu*}^{(2r-1)} = |1+2(\nu^2/n^2)\sum_{k=1}^{\infty}(-1)^{k-1}(k^2-\nu^2/n^2)^{-1}|/|1+\sum_k{}' (1-kn/\nu)^{-2r}|.$$

The sum in the numerator is alternating and has decreasing terms. The sum in the denominator is larger than

$$\begin{aligned}&(1-n/\nu)^{-2r}+(1+n/\nu)^{-2r}\\&=(\nu^2/n^2)^r(1-\nu^2/n^2)^{-2r}[1-\nu/n)^{2r}+(1+\nu/n)^{2r}]\\&>2(\nu^2/n^2)^r(1-\nu^2/n^2)^{-2r}.\end{aligned}$$

Thus,

$$\beta_{\nu*}^{(2r-1)}<\frac{1+2(\nu^2/n^2)(1-\nu^2/n^2)^{-1}}{1+2(\nu^2/n^2)^r(1-\nu^2/n^2)^{-2r}}$$

and if $2\nu^2\leqslant n^2$, then since $(\nu^2/n^2)(1-\nu^2/n^2)^{-1}\leqslant 1$, $\beta_{\nu*}^{(2r-1)}<1+2=3$. If $2\nu^2>n^2$, then

$$(\nu^2/n^2)(1-\nu^2/n^2)^{-1}\leqslant(\nu^2/n^2)^r(1-\nu^2/n^2)^{-r}<(\nu^2/n^2)^r(1-\nu^2/n^2)^{-2r},$$

hence $\beta_{\nu*}^{(2r-1)}<1$. Thus, we have shown

$$\beta_{\nu*}^{(2r-1)}<3,\qquad \nu=1,\ldots,n-1. \tag{6.6}$$

We have proved $|b_\nu^{(s)}(t)|<3(2\pi\nu)^s$ for $\nu=1,2,\ldots,n-1$. Since $b_{\nu+n}=b_\nu$ and $b_{-\nu}=\bar{b}_\nu$, this upper bound is valid for all ν.

In summary, we have

LEMMA 6.2

$$|b_\nu^{(s)}(t)|<3(2\pi\nu)^s,\ -\infty<t<\infty;\ \nu=0,\pm1,\pm2,\ldots;\ s=1,\ldots,2r-1. \tag{6.7}$$

We now investigate the error in approximating $(2\pi i\nu)^s\exp(2\pi i\nu t)$ by $b_\nu^{(s)}(t)$. By (5.9)

$$\begin{aligned}&|(2\pi i\nu)^s e^{2\pi i\nu t}-b_\nu^{(s)}(t)|\leqslant\delta_\nu^{(s)}(2\pi\nu)^s\\&\delta_\nu^{(s)}=\sum_k{}'\,|1-kn/\nu|^{-2r+s}+\sum_k{}'\,|1-kn/\nu|^{-2r}\\&\nu\not\equiv 0\,(\mathrm{mod}\,n);\ s=0,1,\ldots,2r-2.\end{aligned} \tag{6.8}$$

We write, assuming $1\leqslant\nu\leqslant n-1$,

$$\begin{aligned}\delta_\nu^{(s)}&=(n/\nu-1)^{-2r+s}+(n/\nu+1)^{-2r+s}+(n/\nu-1)^{-2r}+(n/\nu+1)^{-2r}\\&\quad+\sum_{k=2}^{\infty}[(kn/\nu-1)^{-2r+s}+(kn/\nu+1)^{-2r+s}+(kn/\nu-1)^{-2r}+(kn/\nu+1)^{-2r}]\end{aligned}$$

and apply inequalities (6.3):

$$\begin{aligned}\delta_\nu^{(s)}&\leqslant n^{-2r+s}[(1-\nu/n)^{-2r+s}+(1+\nu/n)^{-2r+s}+(1-\nu/n)^{-2r}+(1+\nu/n)^{-2r}]\\&\quad+(2r-s-1)^{-1}\nu^{2r-s}[(1-\nu/n)^{-2r+s+1}+(1+\nu/n)^{-2r+s+1}]\\&\quad+(2r-1)^{-1}(\nu/n)^{2r}[(1-\nu/n)^{-2r+1}+(1+\nu/n)^{-2r+1}].\end{aligned}$$

For $2\nu \leqslant n$, this gives

$$\delta_\nu^{(s)} \leqslant n^{-2r+s}[2^{2r-s} + 1 + 2^{2r} + 1 + \nu^{2r-s}(1 + 2^{2r-s-1}) + 2^{-s}\nu^{2r-s}(1 + 2^{2r-1})]$$

from which one concludes easily

$$\delta_\nu^{(s)} \leqslant 2^{2r+2}(\nu/n)^{2r-s}, \qquad 2 \leqslant 2\nu \leqslant n;\ s = 0, 1, \ldots, 2r-2. \tag{6.9}$$

Thus, we have shown

$$|(2\pi i\nu)^s e^{2\pi i\nu t} - b_\nu^{(s)}(t)| \leqslant 2^{2r+2}(2\pi)^s \nu^{2r} n^{s-2r} \tag{6.10}$$
$$\nu = 1, \ldots, [n/2];\ s = 0, 1, \ldots, 2r-2.$$

For $2\nu > n$ we make use of (6.7) and obtain

$$|(2\pi i\nu)^s e^{2\pi i\nu t} - b_\nu^{(s)}(t)| \leqslant |(2\pi i\nu)^s e^{2\pi i\nu t}| + |b_\nu^{(s)}(t)|$$
$$\leqslant (2\pi\nu)^s + 3(2\pi\nu)^s = 4(2\pi)^s(\nu/n)^{s-2r}\nu^{2r} n^{s-2r} \tag{6.11}$$
$$\leqslant 2^{2r+2-s}(2\pi)^s \nu^{2r} n^{s-2r}$$
$$\nu = [n/2]+1, \ldots, n-1;\ s = 0, 1, \ldots, 2r-1.$$

For the case of $s = 2r-1$ we use (5.14), according to which, for $m/n < t < (m+1)/n$

$$|(2\pi i\nu)^{2r-1} e^{2\pi i\nu t} - b_\nu^{(2r-1)}(t)|$$
$$= (2\pi\nu)^{2r-1}|e^{2\pi i\nu t} - \beta_\nu^{(2r-1)} e^{2\pi i\nu(m+1/2)/n}|$$
$$\leqslant (2\pi\nu)^{2r-1}(|e^{2\pi i\nu t} - e^{2\pi i\nu(m+1/2)/n}| + |\beta_\nu^{(2r-1)} - 1|).$$

By (5.14), for $1 \leqslant 2\nu \leqslant n$

$$|\beta_\nu^{(2r-1)} - 1| \leqslant |\textstyle\sum'(-1)^k(1 - kn/\nu)^{-1}|$$
$$\leqslant 2(\nu^2/n^2)(1 - \nu^2/n^2)^{-1}$$
$$\leqslant (4/3)(\nu/n),$$

while the mean-value theorem gives

$$|e^{2\pi i\nu t} - e^{2\pi i\nu(m+1/2)/n}| \leqslant 2\pi\nu/n.$$

We have shown

$$|(2\pi i\nu)^{2r-1} e^{2\pi i\nu t} - b_\nu^{(2r-1)}(t)| \leqslant 8(2\pi)^{2r-1}\nu^{2r} n^{-1}$$
$$\nu = 1, \ldots, [n/2]. \tag{6.12}$$

For $2\nu > n$ we use (6.11) with $s = 2r-1$, and we find the same inequality as (6.12). Clearly, the same inequalities are obtained for negative ν. Altogether, we have proved:

LEMMA 6.3

$$|(2\pi i\nu)^s e^{2\pi i\nu t} - b_\nu^{(s)}(t)| \leqslant 2^{2r+2}(2\pi)^s \nu^{2r} n^{s-2r}$$
$$\nu = 0, \pm 1, \ldots, \pm(n-1);\ s = 0, 1, \ldots, 2r-1. \tag{6.13}$$

It is seen that, for fixed ν, the error in approximating $(2\pi i\nu)^s \exp(2\pi i\nu t)$ by $b_\nu^{(s)}(t)$ is uniformly of order no larger than n^{-2r+s} for $s = 0, 1, \ldots, 2r-1$. That it is exactly of this order is seen by taking $t = 0$ if s is even, $s \geqslant 2$. Then (5.9) gives

$$\begin{aligned} &\lim_{n\to\infty} n^{2r-s}[(2\pi i\nu)^s - b_\nu^{(s)}(0)] \\ &= -2(2\pi i)^s \nu^{2r} \sum_{k=1}^{\infty} k^{-2r+s} \\ &= -2i^s(2\pi)^{2r}\nu^{2r}|B_{2r-s}|/(2r-s)! \qquad s = 2, 4, \ldots, 2r-2. \end{aligned} \tag{6.14}$$

For $s = 0$, the error is of the exact order n^{-2r}. This is seen by taking $t = 1/2n$ in (5.8). We obtain

$$\begin{aligned} &\lim_{n\to\infty} n^{2r}[e^{\pi i\nu/n} - b_\nu(1/2n)] \\ &= 2\nu^{2r} \sum_{k=1}^{\infty} (2k-1)^{-2r} \\ &= \nu^{2r}(2^{2r+1}-2)\,\pi^{2r}|B_{2r}|/(2r)!. \end{aligned} \tag{6.15}$$

Thus, the error in interpolating by periodic $2r$-splines, is of order n^{-2r} even for the function $\cos 2\pi t$.

The order n^{-2r+s} is also obtained for the mean-square error. Indeed, if the Parseval identity is applied to (5.9), one obtains

$$\begin{aligned} &\left\{\int_0^1 |(2\pi i\nu)^s e^{2\pi i\nu t} - b_\nu^{(s)}(t)|^2\,dt\right\}^{1/2} \\ &= (2\pi)^s \nu^{2r} n^{-2r+s} \cdot \left\{\sum_k{}' (k-\nu/n)^{-4r+2s} + (\nu/n)^{2s}\left[\sum_k{}' (k-\nu/n)^{-2r}\right]^2\right\}^{1/2} \Big/ \\ &\left\{1 + (\nu/n)^{2r} \sum_k{}' (k-\nu/n)^{-2r}\right\} \qquad \nu \not\equiv 0 \,(\mathrm{mod}\, n);\ s = 0, 1, \ldots, 2r-1 \end{aligned} \tag{6.16}$$

and from this we get

$$\begin{aligned} &\lim_{n\to\infty} n^{2r-s}\left\{\int_0^1 |(2\pi i\nu)^s e^{2\pi i\nu t} - b_\nu^{(s)}(t)|^2\,dt\right\}^{1/2} \\ &= (2\pi)^s \nu^{2r}\left\{\sum_k{}' k^{-4r+2s}\right\}^{1/2} \\ &= (2\pi\nu)^{2r}\{2|B_{4r-2s}|/(4r-2s)!\}^{1/2} \qquad s = 0, 1, \ldots, 2r-1. \end{aligned} \tag{6.17}$$

We now establish a result that is the analog of Bernstein's inequality on the derivatives of trigonometric polynomials.

LEMMA 6.4. *For any periodic $2r$-spline y with knots at the points m/n ($m = 0, \pm 1, \pm 2, \ldots$) the inequality*

$$\int_0^1 |y^{(s)}(t)|^2\,dt \leqslant 3(2\pi n)^{2s} \int_0^1 |y(t)|^2\,dt$$

$$s = 0, 1, \ldots, 2r-1;\ n = 1, 2, \ldots \tag{6.18}$$

holds.

Proof. If we set $y = \sum_{\nu=0}^{n-1} \eta_\nu b_\nu$, then $y^{(s)} = \sum_{\nu=0}^{n-1} \eta_\nu b_\nu^{(s)}$, and because of the orthogonality of the $b_\nu^{(s)}$, we have

$$\int_0^1 |y^{(s)}(t)|^2\,dt = \sum_{\nu=0}^{n-1} |\eta_\nu|^2 \int_0^1 |b_\nu^{(s)}(t)|^2\,dt. \tag{6.19}$$

By (5.11), for $\nu = 1, 2, \ldots, n-1$,

$$\int_0^1 |b_\nu^{(s)}(t)|^2\,dt = (2\pi n)^{2s} \sum_k (k - \nu/n)^{-4r+2s} \Big/ \Big(\sum_k (k - \nu/n)^{-2r}\Big)^2;$$

hence by (6.4) and (6.6)

$$\int_0^1 |b_\nu^{(s)}(t)|^2\,dt \Big/ \int_0^1 |b_\nu(t)|^2\,dt$$
$$= (2\pi\nu)^{2s} \sum_k (1 - kn/\nu)^{-4r+2s} \Big/ \sum_k (1 - kn/\nu)^{-4r} < 3(2\pi\nu)^{2s}. \tag{6.20}$$

Hence, (6.19) yields

$$\int_0^1 |y^{(s)}(t)|^2\,dt \leqslant 3 \sum_{\nu=0}^{n-1} (2\pi\nu)^{2s} |\eta_\nu|^2 \int_0^1 |b_\nu(t)|^2\,dt$$
$$\leqslant 3(2\pi n)^{2s} \sum_{\nu=0}^{n-1} |\eta_\nu|^2 \int_0^1 |b_\nu(t)|^2\,dt$$
$$= 3(2\pi n)^{2s} \int_0^1 |y(t)|^2\,dt$$

and the lemma is proved.

Since $y^{(p)}$ ($p = 1, \ldots, 2r-2$) is itself a periodic $(2r-p)$-spline with knots at the points m/n [the fact that $2r - p$ may be odd does not affect the argument], we infer from (6.18) the more general inequality

$$\int_0^1 |y^{(s)}(t)|^2\,dt \leqslant 3(2\pi n)^{2s-2p} \int_0^1 |y^{(p)}(t)|^2\,dt, \qquad 0 \leqslant p \leqslant s \leqslant 2r-1. \tag{6.21}$$

We also consider the approximation of $\int_{-\tau}^{\tau} \exp(2\pi i \nu t)\,dt = (1/\pi\nu) \sin 2\pi\nu\tau$ ($\nu = \pm 1, \pm 2, \ldots$) by $\int_{-\tau}^{\tau} b_\nu(t)\,dt$. By (5.8)

$$\int_{-\tau}^{\tau} b_\nu(t)\,dt = (-1/n\pi) \sum_k (k - \nu/n)^{-2r-1} \sin 2\pi(\nu - kn)\,\tau \Big/ \sum_k (k - \nu/n)^{-2r}$$
$$\nu \not\equiv 0 \,(\mathrm{mod}\, n). \tag{6.22}$$

Therefore

$$\int_{-\tau}^{\tau} [e^{2\pi i\nu t} - b_\nu(t)]\,dt$$

$$= \pi^{-1}\left(\frac{\nu}{n}\right)^{2r} \sum_k{}' \left(k - \frac{\nu}{n}\right)^{-2r} [\nu^{-1}\sin 2\pi\nu\tau - (kn-\nu)^{-1}\sin 2\pi(kn-\nu)\,\tau]$$

$$\Big/\left[1 + \left(\frac{\nu}{n}\right)^{2r} \sum{}' \left(k - \frac{\nu}{n}\right)^{-2r}\right] \qquad \nu \not\equiv 0 \,(\mathrm{mod}\, n). \tag{6.23}$$

It follows that

$$\left|(1/\pi\nu)\sin 2\pi\nu\tau - \int_{-\tau}^{\tau} b_\nu(t)\,dt\right|$$

$$\leqslant \pi^{-1}\nu^{2r-1} n^{-2r} \sum_k{}' \left[\left(\frac{\nu}{n}\right)\left(k - \frac{\nu}{n}\right)^{-2r-1} + \left(k - \frac{\nu}{n}\right)^{-2r}\right]$$

$$\Big/\left[1 + \left(\frac{\nu}{n}\right)^{2r} \sum_k{}' \left(k - \frac{\nu}{n}\right)^{-2r}\right], \qquad \nu \not\equiv 0 \,(\mathrm{mod}\, n). \tag{6.24}$$

For $\tau = 1/n$ we obtain the asymptotic evaluation

$$\lim_{n\to\infty} n^{2r+1}\left[(1/\pi\nu)\sin(2\pi\nu/n) - \int_{-1/n}^{1/n} b_\nu(t)\,dt\right]$$

$$= 4\nu^{2r} \sum_{k=1}^{\infty} k^{-2r} = 4(2\pi\nu)^{2r}|B_{2r}|/(2r)!, \qquad \nu \not\equiv 0 \,(\mathrm{mod}\, n). \tag{6.25}$$

7. UNIFORM APPROXIMATION OF $\mathfrak{F}_p$-FUNCTIONS

From here on $\|\ \ \|$ will denote the $\mathscr{L}_\infty$-norm, $\|x\| = \sup_t |x(t)|$. We assume first that x is a trigonometric polynomial

$$x(t) = \sum_{\nu=-N}^{N} \alpha_\nu e^{2\pi i\nu t}. \tag{7.1}$$

Then since S is a linear operator, the interpolating spline Sx is given by

$$Sx(t) = \sum_{\nu=-N}^{N} \alpha_\nu b_\nu(t). \tag{7.2}$$

It follows that the bounds derived for the error $\exp(2\pi i\nu t) - b_\nu(t)$ in Section 6 readily apply to $x - Sx$. Thus, by Lemma 6.3, we have

LEMMA 7.1. *If x is a trigonometric polynomial of degree $\leqslant n-1$, then*

$$\|x^{(s)} - D^s S_r^n x\| \leqslant 2^{2r+2}(2\pi)^s \left(\sum_{\nu=-N}^{N} \nu^{2r}|\alpha_\nu|\right) n^{s-2r}, \qquad s = 0, 1, \ldots, 2r-1. \tag{7.3}$$

Also, by (6.15),

$$\lim_{n\to\infty} n^{2r}[x(1/2n) - S_r^n x(1/2n)]$$
$$= (2^{2r+1} - 2)\pi^{2r}(|B_{2r}|/(2r)!) \sum_{\nu=-N}^{N} \nu^{2r} \alpha_\nu \tag{7.4}$$
$$= (-1)^r 2(1 - 2^{-2r})(|B_{2r}|/(2r)!)\, x^{(2r)}(0).$$

This leads to a formula for $x^{(2r)}(0)$:

$$x^{(2r)}(0) = (-1)^r[(2r)!/2(1 - 2^{-2r})|B_{2r}|] \lim_{n\to\infty} n^{2r}[x(1/2n) - S_r^n x(1/2n)]. \tag{7.5}$$

Another such formula follows from (6.14)

$$x^{(2r)}(0) = i^{2r-s-1}[(2r-s)!/2|B_{2r-s}|] \lim_{n\to\infty} n^{2r-s}[x^{(s)}(0) - (S_r^n x)^{(s)}(0)],$$
$$s = 2, 4, \ldots, 2r-2. \tag{7.6}$$

We remark that since $b_{\nu+kn} = b_\nu$ $(k = \pm 1, \pm 2, \ldots)$, (7.2) may be written as

$$Sx(t) = \sum_{\nu=0}^{n-1} \hat{\xi}_\nu b_\nu(t)$$
$$\hat{\xi}_\nu = \sum_{|\nu+kn|\leqslant N} \alpha_{\nu+kn}. \tag{7.7}$$

Comparison of (7.7) with (5.6) results in well-known formulas for the Fourier coefficients of a trigonometric polynomial in terms of the values on a uniform mesh. By Lemma 6.1 we conclude

$$\|Sx\| \leqslant \sum_{\nu=0}^{n-1} |\hat{\xi}_\nu|$$
$$\leqslant \sum_{\nu=-N}^{N} |\alpha_\nu|. \tag{7.8}$$

Similarly, we have for the derivatives $D^s Sx$:

$$D^s Sx(t) = \sum_{\nu=0}^{n-1} \hat{\xi}_{\nu,N} b_\nu^{(s)}(t), \qquad s = 0, 1, \ldots, 2r-1 \tag{7.9}$$

and by Lemma 6.2,

$$\|D^s Sx\| \leqslant 3(2\pi)^s \sum_{\nu=0}^{n-1} \nu^s |\hat{\xi}_\nu|$$
$$\leqslant 3 \sum_{\nu=-N}^{N} (2\pi|\nu|)^s |\alpha_\nu|, \qquad s = 0, 1, \ldots, 2r-1. \tag{7.10}$$

We extend some of these results to general functions. We consider the linear space of functions

$$x(t) = \sum_{\nu=-\infty}^{\infty} \alpha_\nu e^{2\pi i \nu t} \tag{7.11}$$

with absolutely convergent Fourier series. We define $\sum_\nu |\alpha_\nu|$ as the norm of x, and obtain a Banach space $\mathfrak{F}_0$ (isomorphic to the familiar space l_1). Since the trigonometric polynomials are dense in this space, (7.8) shows that $S = S_r^n$ is a bounded operator from $\mathfrak{F}_0$ to $\mathscr{C}$ (with uniform norm on x); moreover, the bound is uniform with respect to n and r. If $x_N(t)$ denotes the partial sum of (7.11) from $-N$ to N, then $x_N \to x$ in the sense of $\mathfrak{F}_0$ as $N \to \infty$. Therefore, by (7.2) and (7.7)

$$\begin{aligned} S_r^n x(t) &= \lim_{N\to\infty} S_r^n x_N(t) \\ &= \sum_{\nu=-\infty}^{\infty} \alpha_\nu b_\nu(t) \\ &= \sum_{\nu=0}^{n-1} \hat{\xi}_\nu b_\nu(t), \qquad \hat{\xi}_\nu = \sum_{k=-\infty}^{\infty} \alpha_{\nu+kn}, \end{aligned} \tag{7.12}$$

where the limit of the infinite sum is (7.12) is uniform with respect to t, n, and r.

If x has a Fourier expansion (7.11) with $\sum_\nu |\nu|^p |\alpha_\nu| < \infty$ for some p, $0 \leqslant p \leqslant 2r$ (p need not be an integer), then we may consider $\sum_\nu |\nu|^p |\alpha_\nu|$ as the norm of x (for $p > 0$ this is a true norm only if functions differing by a constant are identified), and this results again in a Banach space $\mathfrak{F}_p$. We set

$$\|x\|_{\mathfrak{F}_p} = \sum_{\nu=-\infty}^{\infty} (2\pi|\nu|)^p |\alpha_\nu|, \qquad 0 \leqslant p. \tag{7.13}$$

Clearly, if p is an integer, then $\|x\|_{\mathfrak{F}_p} = \|x^{(p)}\|_{\mathfrak{F}_0}$. On this space not only S, but $DS, \ldots, D^p S$ as well, are bounded transformations to $\mathscr{C}$, as we see from (7.10). We may also say that S is a bounded transformation from $\mathfrak{F}_s$ to $\mathscr{C}_s$ (with uniform norm on the sth derivative of x).

The results of (7.8) and (7.10) are summarized in

LEMMA 7.2

$$\|S_r^n x\| \leqslant \|x\|_{\mathfrak{F}_0}, \qquad x \in \mathfrak{F}_0 \tag{7.14a}$$

$$\|D^s S_r^n x\| \leqslant 3\|x\|_{\mathfrak{F}_s}, \qquad x \in \mathfrak{F}_s;\ s = 0, 1, \ldots, 2r-1. \tag{7.14b}$$

It now follows that for $x \in \mathfrak{F}_p$ $(0 \leqslant p \leqslant 2r)$

$$\begin{aligned} D^s S_r^n x(t) &= \sum_{\nu=-\infty}^{\infty} \alpha_\nu b_\nu^{(s)}(t) \\ &= \sum_{\nu=0}^{n-1} \hat{\xi}_\nu b_\nu^{(s)}(t), \qquad s = 0, 1, \ldots, [p] \end{aligned} \tag{7.15}$$

where the limit of the infinite sum in (7.15) is uniform with respect to t, n, and r.

The following error estimates are based on Lemmas 7.1 and 7.2. We obtain from these, for $x \in \mathfrak{F}_p$ $(0 \leqslant p \leqslant 2r)$

$$\begin{aligned}\|x^{(s)} - D^s Sx\| &\leqslant \|x_N^{(s)} - D^s Sx_N\| + \|x^{(s)} - x_N^{(s)}\| + \|D^s Sx - D^s Sx_N\| \\ &\leqslant 2^{2r+2}(2\pi)^s n^{s-2r} \sum_{|\nu|\leqslant N} \nu^{2r}|\alpha_\nu| + \sum_{|\nu|>N} (2\pi|\nu|)^s|\alpha_\nu| \\ &\quad + 3 \sum_{|\nu|>N} (2\pi|\nu|)^s|\alpha_\nu| \\ &\leqslant 2^{2r+2}(2\pi)^s \left\{N^{2r-p} n^{s-2r} \sum_{|\nu|\leqslant N} |\nu|^p|\alpha_\nu| + N^{s-p} \sum_{|\nu|>N} |\nu|^p|\alpha_\nu|\right\}\end{aligned}$$

$$s = 0, 1, \ldots, [p]. \qquad (7.16)$$

For $s = p \leqslant 2r - 1$, we take $N = [n^{1/2}]$ in (7.16) and obtain

$$\|x^{(p)} - D^p Sx\| \leqslant 2^{2r+2}(2\pi)^p \left\{n^{p/2-r} \sum_{|\nu|\leqslant N} |\nu|^p|\alpha_\nu| + \sum_{|\nu|>N} |\nu|^p|\alpha_\nu|\right\}. \quad (7.17)$$

Clearly, (7.17) yields

$$\|x^{(p)} - D^p S_r^n x\| = o(1) \quad \text{as } n \to \infty,$$
$$x \in \mathfrak{F}_p, \qquad p = 0, 1, \ldots, 2r-1. \qquad (7.18)$$

In particular, *the spline interpolants $S_r^n x$ converge to the function x uniformly if $x \in \mathfrak{F}_0$* (i.e. $\sum_\nu |\alpha_\nu| < \infty$).

If $s < p$, then we take $N = n - 1$ in (7.16) and obtain

$$\|x^{(s)} - D^s S_r^n x\| \leqslant 2^{2r+2}(2\pi)^{s-p}\|x\|_{\mathfrak{F}_p} n^{s-p}$$
$$x \in \mathfrak{F}_p, \quad 0 \leqslant s < p \leqslant 2r. \qquad (7.19)$$

Thus, $x^{(s)}$ is approximated by $D^s S_r^n x$ with an error of order $O(n^{s-p})$ in the class $\mathfrak{F}_p$, and an explicit bound on the coefficient of n^{s-p} is established. Remarkable is that if $x \in \mathfrak{F}_{2r}$, then even the discontinuous (piecewise constant) $D^{2r-1} S_r^n x$ converge to $x^{(2r-1)}$, with an error term of order $O(n^{-1})$.

For $x \in \mathfrak{F}_{2r}$, the error in the approximation of $x^{(s)}$ is of order $O(n^{s-2r})$, just as for trigonometric polynomials. That the error cannot be of higher order is clear from the fact that it is of the precise order $O(n^{s-2r})$ for $x(t) = \cos 2\pi t$ [see (6.14)]. Moreover, we can extend (7.4) to the function x in $\mathfrak{F}_{2r}$. We write

$$\begin{aligned}n^{2r}[x(1/2n) - Sx(1/2n)] &= n^{2r}[x_N(1/2n) - Sx_N(1/2n)] \\ &\quad + n^{2r}[(x - x_N)(1/2n) - S(x - x_N)(1/2n)]. \quad (7.20)\end{aligned}$$

By (7.19) we have

$$n^{2r}|(x - x_N)(1/2n) - S_r^n(x - x_N)(1/2n)| \leqslant 2^{2r+2}(2\pi)^{-2r}\|x - x_N\|_{\mathfrak{F}_{2r}} \quad (7.21)$$

and this can be made arbitrarily small, independent of n, by choosing N sufficiently large. Thus, (7.20) in conjunction with (7.4) and (7.21) gives

$$\lim_{n\to\infty} n^{2r}[x(1/2n) - S_r^n x(1/2n)] = (-1)^r 2(1 - 2^{-2r})(|B_{2r}|/(2r)!)\, x^{(2r)}(0) \quad (7.22)$$

for every $x \in \mathfrak{F}_{2r}$. Eq. (7.22) may be considered a formula for $x^{(2r)}(0)$:

$$x^{(2r)}(0) = (-1)^r[(2r)!/2(1-2^{-2r})|B_{2r}|] \lim_{n\to\infty} n^{2r}[x(1/2n) - S_r{}^n x(1/2n)], \quad x \in \mathfrak{F}_{2r}. \tag{7.23}$$

In the same way (7.6) is extended, and gives

$$x^{(2r)}(0) = i^{2r-s-1}[(2r-s)!/2|B_{2r-s}|] \lim_{n\to\infty} n^{2r-s}[x^{(s)}(0) - D^s S_r{}^n x(0)],$$
$$s = 2, 4, \ldots, 2r-2, \quad x \in \mathfrak{F}_{2r}. \tag{7.24}$$

From (7.23) we conclude that if $x \in \mathfrak{F}_{2r}$ and $x(1/2n) - S_r{}^n(1/2n) = o(n^{-2r})$ as $n \to \infty$, then $x^{(2r)}(0) = 0$. Using only the sequence $n = 2^m$ ($m = 0, 1, 2, \ldots$), we may also conclude from (7.23) that if $x \in \mathfrak{F}_{2r}$ and $\|x - S_n{}^r x\| = o(n^{-2r})$ as $n \to \infty$, then $x^{(2r)}(k \cdot 2^{-m}) = 0$ for each m and integer k. Since $x^{(2r)}$ is continuous, this implies $x^{(2r)} = 0$, hence x is the constant function. We have proved:

If $x \in \mathfrak{F}_{2r}$ and $\|x - S_n{}^r x\| = o(n^{-2r})$, then x is constant.

In similar fashion we conclude from (7.24):

If $x \in \mathfrak{F}_{2r}$ and $\|D^s x - D^s S_r{}^n x\| = o(n^{s-2r})$ for some $s = 0, 1, \ldots, 2r-1$, then x is constant.

We summarize several of these results in

THEOREM 7.1. *Suppose $S_r{}^n x(t)$ is the periodic $2r$-spline ($r \geqslant 1$) that interpolates the function $x(t)$ at the knots m/n ($m = 0, \pm 1, \pm 2, \ldots$). If s is one of the integers $0, 1, \ldots, 2r-1$ and if $x \in \mathfrak{F}_p$ for some p, $s \leqslant p \leqslant 2r$, then $\|x^{(s)} - (S_r{}^n x)^{(s)}\| = O(n^{-p+s})$ [$o(1)$ if $s = p$] as $n \to \infty$. In particular, if $x \in \mathfrak{F}_{2r}$, then $\|x^{(s)} - (S_r{}^n x)^{(s)}\| = O(n^{-2r+s})$, and if $\|x^{(s)} - (S_r{}^n x)^{(s)}\| = o(n^{-2r+s})$ for some s, $0 \leqslant s \leqslant 2r-1$, then x is constant.*

The special case $p = 2r-2$ (with the weaker hypothesis $x \in \mathscr{C}_{2r-2}$ in place of $x \in \mathfrak{F}_{2r-2}$ and with a more general sequence of meshes) appears in [*5*, Theorem 4]. However, the conclusion there is only $x^{(s)} - (S_r{}^n x)^{(s)} = o(1)$ for $s = 0, 1, \ldots, 2r-2$. In the same paper the case $p = r$ appears (again $x \in \mathscr{C}_r$ in place of $x \in \mathfrak{F}_r$, and a more general sequence of meshes is considered), and the conclusion is $x^{(s)} - (S_r{}^n x)^{(s)} = o(1)$ only for $s = 0, 1, \ldots, r-1$. There are more precise results in [*7*], however this source was not available at the time this article was written. Related results are also found in [*10*].

8. Mean-Square Approximation of $\mathscr{W}_p$ Functions

We now consider functions $x(t)$ with Fourier expansion $\sum_\nu \alpha_\nu \exp(2\pi i \nu t)$ for which

$$\sum_{\nu=-\infty}^{\infty} |\nu|^{2p} |\alpha_\nu|^2 < \infty. \tag{8.1}$$

The number p need not be an integer, but we do assume $p > \frac{1}{2}$. We call the space of these functions $\mathscr{W}_p$, and provide it with the norm

$$\|x\|_{\mathscr{W}_p} = \left\{ \sum_{\nu=-\infty}^{\infty} (2\pi|\nu|)^{2p} |\alpha_\nu|^2 \right\}^{1/2}, \tag{8.2a}$$

which clearly comes from an inner product. $\mathscr{W}_p$ is a Hilbert space. In particular, if p is an integer, then $\mathscr{W}_p$ is the Sobolev space of periodic functions x that have derivatives x', x'', ..., $x^{(p-1)}$, with $x^{(p-1)}$ absolutely continuous and the Lebesgue derivative $x^{(p)}$ square-integrable. The norm defined above is also given by

$$\|x\|_{\mathscr{W}_p} = \left\{ \int_0^1 |x^{(p)}(t)|^2 \, dt \right\}^{1/2} = \|x^{(p)}\|_2 \tag{8.2b}$$

if p is an integer. $\| \; \|_2$ will denote the $\mathscr{L}_2$ norm from here on. As before, functions differing by a constant are identified [or $\alpha_0 = \int_0^1 x(t)\,dt = 0$ is assumed for each x].

Since $\sum_\nu |\nu|^{2p} |\alpha_\nu|^2 < \infty$ implies

$$\sum_{\nu>0} \nu^{p-1/2-\epsilon} |\alpha_\nu| \leqslant \left\{ \sum_{\nu>0} \nu^{2p} |\alpha_\nu|^2 \sum_{\nu>0} \nu^{-1-2\epsilon} \right\}^{1/2} < \infty, \tag{8.3}$$

we conclude $\mathfrak{F}_p \subset \mathscr{W}_p \subset \mathfrak{F}_{p-1/2-\epsilon}$ for every $\epsilon > 0$. It then follows from Theorem 7.1 that $\|x^{(s)} - D^s S_r^n x\| = O(n^{s-p+1/2+\epsilon})$ for $x \in \mathscr{W}_p$ and $s < p - \frac{1}{2}$. We will show that this error is actually $O(n^{s-p+1/2})$ and that the root mean-square error $\|x^{(s)} - D^s S_r^n x\|_2$ is $O(n^{s-p})$.

The function $(2\pi i\mu)^s \exp(2\pi i\mu t) - b_\mu^{(s)}(t)$ is orthogonal (in $\mathscr{L}_2$) to the function $(2\pi i\nu)^s \exp(2\pi i\nu t) - b_\nu^{(s)}(t)$ if μ, ν are integers not congruent (mod n). Therefore, if

$$x(t) = \sum_{|\nu| \leqslant N} \alpha_\nu e^{2\pi i\nu t}$$

is a trigonometric polynomial of degree $N \leqslant [n/2]$ (if $N = n/2$, it is assumed that either $\alpha_N = 0$ or $\alpha_{-N} = 0$), then by Lemma 6.3

$$\begin{aligned} \|x^{(s)} - D^s S_r^n x\|_2^2 &\leqslant \sum_{|\nu| \leqslant N} |\alpha_\nu|^2 \int_0^1 |(2\pi i\nu)^s e^{2\pi i\nu t} - b_\nu^{(s)}(t)|^2 \, dt \\ &\leqslant 2^{4r+4} (2\pi)^{2s} \left(\sum_{|\nu| \leqslant N} |\nu|^{4r} |\alpha_\nu|^2 \right) n^{2s-4r}. \end{aligned} \tag{8.4}$$

We formulate this as

LEMMA 8.1. *If x is a trigonometric polynomial of degree $N \leqslant [n/2]$, then*

$$\|x^{(s)} - D^s S_r^n x\|_2 \leqslant 2^{2r+2} (2\pi)^{s-2r} \|x\|_{\mathscr{W}_{2r}} n^{s-2r}, \qquad s = 0, 1, \ldots, 2r-1. \tag{8.5}$$

The order of this error bound is sharp. Indeed, (6.17) gives for any trigonometric polynomial x

$$\lim_{n\to\infty} n^{2r-s}\|x^{(s)} - D^s S_r{}^n x\|_2 = \{2|B_{4r-2s}|/(4r-2s)!\}^{1/2}\|x\|_{\mathscr{W}_{2r}}, \quad s = 0, 1, \ldots, 2r-1. \tag{8.6}$$

The spline interpolant S may be considered as a linear transformation from $\mathscr{W}_p$ to $\mathscr{W}_s$. We show that this transformation is bounded if $s < p - \frac{1}{2}$.

LEMMA 8.2. *If* $x \in \mathscr{W}_p$ $(p > \frac{1}{2})$,

$$x(t) = \sum_{|\nu| \geq N} \alpha_\nu e^{2\pi i \nu t} \quad (N \geq 1),$$

then

$$\|S_r{}^n x\|_{\mathscr{W}_s}^2 \leq 9(2\pi)^{2s-2p} 2n^{-1}(2p-2s-1)^{-1} N^{-2p+2s+1}\|x\|_{\mathscr{W}_p}^2, \quad s < p - \tfrac{1}{2}. \tag{8.7}$$

Proof. Since $\mathscr{W}_p \subset \mathfrak{F}_0$ for $p > \frac{1}{2}$, the Fourier series (7.11) of x converges absolutely, and by (7.9) we have

$$D^s Sx = \sum_\nu \hat{\xi}_\nu b_\nu^{(s)}, \qquad \xi_\nu = \sum_{k=-\infty}^{\infty} \alpha_{\nu+kn}, \tag{8.8}$$

where we let ν range from $-[(n-1)/2]$ to $[n/2]$ instead of from 0 to $n-1$. Then, by Lemma 6.2,

$$\|D^s Sx\|_2{}^2 = \sum_\nu |\hat{\xi}_\nu|^2 \|b_\nu^{(s)}\|_2{}^2$$
$$\leq 9(2\pi)^{2s} \sum_\nu \nu^{2s}|\hat{\xi}_\nu|^2, \qquad s = 0, 1, \ldots, 2r-1. \tag{8.9}$$

By the Schwarz inequality,

$$|\sum \alpha_{\nu+kn}|^2 \leq \sum |\nu+kn|^{-2p+2s} \sum |\nu+kn|^{2p-2s}|\alpha_{\nu+kn}|^2. \tag{8.10}$$

Using the simple inequality

$$\sum_{|\nu+kn|\geq N} |\nu+kn|^{-2p+2s} \leq 2n^{-1}(2p-2s-1)^{-1} N^{-2p+2s+1}$$

and the fact that $|\nu| \leq |\nu+kn|$ for the values of ν employed, we obtain

$$\sum_\nu \nu^{2s}|\sum_k \alpha_{\nu+kn}|^2 \leq 2n^{-1}(2p-2s-1)^{-1} N^{-2p+2s+1} \sum_{\mu=-\infty}^{\infty} |\mu|^{2p}|\alpha_\mu|^2. \tag{8.11}$$

(8.11) together with (8.9) yield (8.7).

Now assume $x \in \mathscr{W}_p$ $(p > \frac{1}{2})$ and

$$x_N(t) = \sum_{|\nu|\leq N} \alpha_\nu e^{2\pi i \nu t}, \quad N \leq [n/2].$$

Then we obtain, using Lemmas 8.1 and 8.2

$$\|x^{(s)} - D^s Sx\|_2 \leqslant \|x_N^{(s)} - D^s Sx_N\|_2 + \|x^{(s)} - x_N^{(s)}\|_2 + \|D^s Sx - D^s Sx_N\|_2$$
$$\leqslant 2^{2r+2}(2\pi)^s \left\{\sum_{|\nu|\leqslant N} \nu^{4r}|\alpha_\nu|^2\right\}^{1/2} n^{s-2r} + \left\{\sum_{|\nu|>N} (2\pi\nu)^{2s}|\alpha_\nu|^2\right\}^{1/2}$$
$$+ 3(2\pi)^{s-p}\left\{2n^{-1}(2p-2s-1)^{-1} N^{-2p+2s+1} \sum_{|\nu|>N} (2\pi\nu)^{2p}|\alpha_\nu|^2\right\}^{1/2}. \tag{8.12}$$

We choose $N = [n/2]$ and find

$$\|x^{(s)} - D^s Sx\|_2 \leqslant 2^{p+2}(2\pi)^{s-p}\left\{\sum_{|\nu|\leqslant N} |2\pi\nu|^{2p}|\alpha_\nu|^2\right\}^{1/2} n^{s-p}$$
$$+ 2^{p-s}(2\pi)^{s-p}\left\{\sum_{|\nu|>N} |2\pi\nu|^{2p}|\alpha_\nu|^2\right\}^{1/2} n^{s-p}$$
$$+ 3(2\pi)^{s-p} 2^{p-s}(2p-2s-1)^{-1/2}\left\{\sum_{|\nu|>N} |2\pi\nu|^{2p}|\alpha_\nu|^2\right\}^{1/2} n^{s-p}. \tag{8.13}$$

Therefore, we have proved

$$\|x^{(s)} - D^s S_r^n x\|_2 \leqslant (2\pi)^{s-p} 2^{p+2}[1 + 3(2p-2s-1)^{-1/2}] n^{s-p}\|x\|_{\mathscr{W}_p},$$
$$s + \tfrac{1}{2} < p \leqslant 2r. \tag{8.14}$$

Thus, $x^{(s)}$ is approximated [in the square-mean] by $D^s S_r^n x$ with an error of order $O(n^{s-p})$ in the class $\mathscr{W}_p$ $(p > s + \frac{1}{2})$, and an explicit bound on the coefficient of n^{s-p} is established. For $x \in \mathscr{W}_{2r}$, the error in the approximation of $x^{(s)}$ is of order n^{s-2r}, just as for trigonometric polynomials. That the error cannot be of higher order is shown by extending equation (8.6) to general functions in $\mathscr{W}_{2r}$. By the triangle inequality we have

$$|n^{2r-s}\|x^{(s)} - D^s Sx\|_2 - n^{2r-s}\|x_N^{(s)} - D^s Sx_N\|_2|$$
$$\leqslant n^{2r-s}\|(x - x_N) - D^s S(x - x_N)\|_2. \tag{8.15}$$

By (8.14) we have

$$n^{2r-s}\|(x - x_N)^{(s)} - D^s S(x - x_N)\|_2 \leqslant C\|x - x_N\|_{\mathscr{W}_{2r}} \tag{8.16}$$

with a constant C that is independent of n and N. (8.16) can be made arbitrarily small by choosing N sufficiently large (independent of n). Thus, with the use of (8.6) and (8.16), (8.15) yields

$$\lim_{n\to\infty} n^{2r-s}\|x^{(s)} - D^s S_r^n x\|_2 = \{2|B_{4r-2s}|/(4r-2s)!\}^{1/2}\|x\|_{\mathscr{W}_{2r}}$$
$$s = 0, 1, \ldots, 2r-1 \tag{8.17}$$

for any function x in $\mathscr{W}_{2r}$. In particular, this implies

If $x \in \mathscr{W}_{2r}$ and $\|x^{(s)} - (S_r^n)^{(s)} x\|_2 = o(n^{s-2r})$ for some $s = 0, 1, \ldots, 2r-1$, then x is constant

We summarize some of these results in

THEOREM 8.1. *Suppose $S_r^n x(t)$ is the periodic $2r$-spline ($r \geqslant 1$) that interpolates the function $x(t)$ at the knots m/n ($m = 0, \pm 1, \pm 2, \ldots$). If s is one of the integers $0, 1, \ldots, 2r-1$ and if $x \in \mathscr{W}_p$ for some p, $s + \frac{1}{2} < p \leqslant 2r$, then*

$$\left\{\int_0^1 |x^{(s)}(t) - (S_r^n x)^{(s)}(t)|^2 \, dt\right\}^{1/2} = O(n^{s-p}) \quad \text{as } n \to \infty.$$

In particular, if $x \in \mathscr{W}_{2r}$, then

$$\left\{\int_0^1 |x^{(s)}(t) - (S_r^n x)^{(s)}(t)|^2 \, dt\right\}^{1/2} = O(n^{-2r+s}),$$

and if this error is of order $o(n^{-2r-s})$ for some s, $0 \leqslant s \leqslant 2r-1$, then x is constant.

Similar results for the cases $p = r$ and $p = 2r$ have also been obtained (for more general meshes and more general types of splines) in [*8*, Theorems 7 and 13]. The conclusion of that paper concerning the case $p = 2r$ is weaker, inasmuch as $O(n^{s-2r})$ is replaced by $O(n^{s-2r+1/2})$, for $s = r+1, \ldots, 2r-1$. Related results are also found in [*7*]; however, this source was not available when this article was written.

The case $p = r$ deserves special attention. It is well known (see [*1*], p. 133; [*3*] and [*5*]), that among all functions $y \in \mathscr{W}_r$ that interpolate a given function $x \in \mathscr{W}_r$ at the points m/n ($m = 0, \pm 1, \pm 2, \ldots$), the $2r$-spline $y = S_r^n x$ attains the minimal value of $\int_0^1 |y^{(r)}(t)|^2 dt$ and that $\int_0^1 (S_r^n x)^{(r)}(t) \overline{x_0^{(r)}(t)} \, dt = 0$ for any function $x_0 \in \mathscr{W}_r$ for which $x_0(m/n) = 0$ ($m = 0, \pm 1, \ldots$). Therefore,

$$\|D^r S_r^n x\|_2 \leqslant \|x\|_{\mathscr{W}_r}, \qquad x \in \mathscr{W}_r \tag{8.18a}$$

and

$$\|x^{(r)} - D^r S_r^n x\|_2{}^2 = \|x\|_{\mathscr{W}_r}^2 - \|S_r^n x\|_{\mathscr{W}_r}^2, \qquad x \in \mathscr{W}_r. \tag{8.18b}$$

We may now state

THEOREM 8.2. *Suppose $S_r^n x(t)$ is the periodic $2r$-spline ($r \geqslant 1$) that interpolates the function $x(t)$ at the knots m/n ($m = 0, \pm 1, \pm 2, \ldots$). If $x \in \mathscr{W}_r$, then*

$$\int_0^1 |x^{(r)}(t) - (S_r^n x)^{(r)}(t)|^2 \, dt = \int_0^1 |x^{(r)}(t)|^2 \, dt - \int_0^1 |(S_r^n x)^{(r)}(t)|^2 \, dt$$
$$= o(1) \quad \text{as } n \to \infty. \tag{8.19}$$

Proof. By (8.12), using $N = [n^{1/2}]$ (which is $<[n/2]$ for $n \geqslant 6$), we have, for n sufficiently large

$$\|x^{(r)} - D^r S_r^n x\|_2 \leqslant 2^{2r+2} n^{-r/2} \left\{\sum_{|\nu| \leqslant N} (2\pi\nu)^{2r} |\alpha_\nu|^2\right\}^{1/2} + 2\left\{\sum_{|\nu| > N} (2\pi\nu)^{2r} |\alpha_\nu|^2\right\}^{1/2}$$
$$= o(1) \quad \text{as } n \to \infty. \tag{8.20}$$

This result is remarkable since the approximated function is $x^{(r)}$, which is an arbitrary function in $\mathscr{L}_2$. This case is dealt with in [*8*, Theorem 7], but the conclusion there is only $\|x^{(r)} - D^r S_r^n x\|_2 = O(1)$ as $n \to \infty$.

By Theorem 8.1, $\|x - S_r^n x\|_2 = O(n^{-p})$ if $x \in \mathscr{W}_p$ $(p > \frac{1}{2})$. The converse of this statement is not true. However, we now prove a result that is very close to a converse.

THEOREM 8.3. *Suppose $S_r^n x$ is the periodic 2r-spline ($r \geqslant 1$) that interpolates the square-integrable function x at the points m/n ($m = 0, \pm 1, \pm 2, \ldots$), and $\left\{\int_0^1 |x(t) - S_r^n x(t)|^2 dt\right\}^{1/2} = O(n^{-q})$ for some $1 < q \leqslant 2r$ and $n = 1, 2, 4, 8, \ldots$. Then x is equal almost everywhere to a function $x_* \in \mathscr{W}_p$, where p is the largest integer smaller than q.*

Proof. If

$$\|x - S^n x\|_2 \leqslant Cn^{-q}, \quad n = 1, 2, 4, \ldots, \tag{8.21}$$

then

$$\|S^n x - S^{2n} x\|_2 \leqslant 2Cn^{-q}, \qquad n = 1, 2, 4, \ldots. \tag{8.22}$$

The function $S^n x - S^{2n} x$ is a $2r$-spline with knots at the points $m/2n$ ($m = 0, \pm 1, \pm 2, \ldots$). By Lemma 6.4, for $s = 0, 1, \ldots, 2n - 1$

$$\|D^s S^n x - D^s S^{2n} x\|_2 \leqslant C_1 n^{s-q}, \qquad n = 1, 2, 4, \ldots \tag{8.23}$$

where $C_1 = (12)^{1/2}(4\pi)^s C$. Thus, if $m = 2^k n$ (k a positive integer), then

$$\begin{aligned} \|D^s S^n x - D^s S^m x\|_2 &\leqslant \sum_{l=0}^{k-1} \|D^s S^{2^l n} x - D^s S^{2^{l+1} n} x\|_2 \\ &\leqslant C_1 \sum_{l=0}^{k-1} (2^l n)^{s-q} \\ &< C_1 n^{s-q}/(1 - 2^{s-q}). \end{aligned} \tag{8.24}$$

It follows that, for $s = 0, 1, \ldots, p$ the sequence of functions $D^s S^n x$ ($n = 1, 2, 4, \ldots$) converges (in $\mathscr{L}_2$), while by hypothesis the sequence $S^n x$ converges to x. Since $\mathscr{W}_p$ is complete, the conclusion of the theorem follows.

It is not true that $\|x - S_r^n x\|_2 = O(n^{-2r})$ ($n = 1, 2, 4, \ldots$) implies $x \in \mathscr{W}_{2r}$. This is seen by taking for x a $2r$-spline that has knots at the points $m \cdot 2^{-k}$ ($m = 0, \pm 1, \pm 2, \ldots$), for some positive integer k, with $x^{(2r-1)}$ discontinuous at some of these knots. Then $S^n x = x$ for $n \geqslant 2^k$, but $x \notin \mathscr{W}_{2r}$.

9. UNIFORM APPROXIMATION OF $\mathscr{W}_p$ FUNCTIONS

As in the preceding section the functions to be approximated by $2r$-splines are periodic and in $\mathscr{W}_p$ for some $p > \frac{1}{2}$. As before, $\| \;\|$ denotes the $\mathscr{L}_\infty$-norm,

$\| \|_{\mathscr{W}_p}$ the norm in $\mathscr{W}_p$. The spline interpolant S, considered as a linear transformation from $\mathscr{W}_p$ to $\mathscr{C}_s$ ($s < p - \frac{1}{2}$) is bounded. A bound for this transformation is given in

LEMMA 9.1. *If* $x \in \mathscr{W}_p$ $(p > \frac{1}{2})$,

$$x(t) = \sum_{|\nu| \geqslant N} \alpha_\nu e^{2\pi i \nu t} \quad (N \geqslant 1),$$

then

$$\|D^s S_r^n x\| \leqslant 3(2\pi)^{s-p}\{2n^{-1}(2p-2s-1)^{-1} N^{-2p+2s+1}\}^{1/2} \|x\|_{\mathscr{W}_p},$$
$$s < p - \tfrac{1}{2}. \tag{9.1}$$

Proof. We proceed as in the proof of Lemma 8.2, with (8.8) replaced by

$$\|D^s Sx\| \leqslant \sum_\nu |\hat{\xi}_\nu| \, \|b_\nu^{(s)}\|$$
$$\leqslant 3(2\pi)^s \sum_\nu |\nu|^s |\hat{\xi}_\nu|, \qquad s = 0, 1, \ldots, 2r-1. \tag{9.2}$$

By (8.9) and (8.10) we have

$$|\hat{\xi}_\nu| \leqslant \{2n^{-1}(2p-2s-1)^{-1} N^{-2p+2s+1} \sum |\nu + kn|^{2p-2s} |\alpha_{\nu+kn}|^2\}^{1/2};$$

hence

$$\sum_\nu |\nu|^s |\hat{\xi}_\nu| \leqslant \{2n^{-1}(2p-2s-1)^{-1} N^{-2p+2s+1} \sum_{\mu=-\infty}^{\infty} |\mu|^{2p} |\alpha_\mu|^2\}^{1/2},$$

so that (9.2) yields (9.1).

If we apply the Schwarz inequality to the finite sum in (7.3), we obtain for

$$x(t) = \sum_{|\nu| \leqslant N} \alpha_\nu e^{2\pi i \nu t} \quad (N \leqslant n-1)$$

$$\|x^{(s)} - D^s S_r^n x\| \leqslant 2^{2r+2}(2\pi)^{s-p} \left\{ \sum_{|\nu| \leqslant N} |\nu|^{4r-2p} \sum_{|\nu| \leqslant N} |\nu|^{2p} |\alpha_\nu|^2 \right\}^{1/2} n^{s-2r} \tag{9.3}$$

and since $\sum |\nu|^{4r-2p} \leqslant (2N+1) N^{4r-2p}$,

$$\|x^{(s)} - D^s S_r^n x\| \leqslant 2^{2r+2}(2\pi)^s \left\{ (2N+1) N^{4r-2p} \sum_{|\nu| \leqslant N} |\nu|^{2p} |\alpha_\nu|^2 \right\}^{1/2} n^{s-2r}$$
$$p \geqslant 0; \; s = 0, 1, \ldots, 2r-1. \tag{9.4}$$

Using (9.4) and Lemma 9.1, we find for $x \in \mathscr{W}_p$ $(p > \frac{1}{2})$, with x_N the partial Fourier sum as before,

$$\|x^{(s)} - D^s Sx\| \leqslant \|x_N^{(s)} - D^s Sx_N\| + \|x^{(s)} - x_N^{(s)}\| + \|D^s Sx - D^s Sx_N\|$$
$$\leqslant 2^{2r+2}(2\pi)^{s-p}\{(2N+1) N^{4r-2p} \sum_{|\nu| \leqslant N} |\nu|^{2p} |\alpha_\nu|^2\}^{1/2} n^{s-2r}$$
$$+ \sum_{|\nu| > N} |2\pi\nu|^s |\alpha_\nu| + 3(2\pi)^{s-p}\{2n^{-1}(2p-2s-1)^{-1} \; N^{-2p+2s+1}$$
$$\sum_{|\nu| > N} |2\pi\nu|^{2p} |\alpha_\nu|^2\}^{1/2}. \tag{9.5}$$

We take $N = [n/2] - 1$, and obtain

$$\|x^{(s)} - D^s Sx\| \leqslant 2^{p-2}(2\pi)^{s-p}\left\{\sum_{|\nu|\leqslant N} |\nu|^{2p}|\alpha_\nu|^2\right\}^{1/2} n^{s-p+1/2}$$

$$+ 2^{p-s}(2\pi)^{s-p}\left\{(2p-2s-1)^{-1} \sum_{|\nu|>N} |2\pi\nu|^{2p}|\alpha_\nu|^2\right\}^{1/2} n^{s-p+1/2}$$

$$+ 3\cdot 2^{p-s}(2\pi)^{s-p}\left\{(2p-2s-1)^{-1} \sum_{|\nu|>N} |2\pi\nu|^{2p}|\alpha_\nu|^2\right\}^{1/2} n^{s-p+1/2}, \qquad (9.6)$$

where we have used the inequality

$$\sum_{|\nu|>N} |\nu|^s|\alpha_\nu| \leqslant \left\{\sum_{|\nu|>N} |\nu|^{2s-2p}\right\}^{1/2}\left\{\sum_{|\nu|>N} |\nu|^{2p}|\alpha_\nu|^2\right\}^{1/2}$$

$$\leqslant 2^{-s+p}\left\{(2p-2s-1)^{-1} \sum_{|\nu|>N} |\nu|^{2p}|\alpha_\nu|^2\right\}^{1/2} n^{s-p+1/2} \qquad (9.7)$$

The final result derived from (9.6) is

$$\|x^{(s)} - D^s S_r^n x\| \leqslant (2\pi)^{s-p}\, 2^{p+2}\left(\frac{2p-1}{2p-2s-1}\right)^{1/2} n^{s-p+1/2}\|x\|_{\mathscr{W}_p},$$

$$s + \tfrac{1}{2} < p \leqslant 2r. \qquad (9.8)$$

Thus, $x^{(s)}$ is approximated uniformly by $D^s S_r^n x$ with an error of order $O(n^{s-p+1/2})$ in the class $\mathscr{W}_p$ $(p > s + \frac{1}{2})$, and an explicit bound on the coefficient of $n^{s-p+1/2}$ is established. For $x \in \mathscr{W}_r$ and s one of the numbers $0, 1, \ldots, r-1$, the error is of order $O(n^{-r+s+1/2})$, and that this is the best possible, is proved below (see Theorem 11.2). We state the result in

THEOREM 9.1. *Suppose $S_r^n x(t)$ is the periodic 2r-spline $(r \geqslant 1)$ that interpolates the function $x(t)$ at the knots m/n $(m = 0, \pm 1, \pm 2, \ldots)$. If s is one of the integers $0, 1, \ldots, 2r-1$ and if $x \in \mathscr{W}_p$ for some p, $s + \frac{1}{2} < p \leqslant 2r$, then $\|x^{(s)} - (S_r^n x)^{(s)}\| = O(n^{s-p+1/2})$ as $n \to \infty$.*

In [6, Theorem 3] it is proved that if $x \in \mathscr{W}_r$, then $|x^{(s)}(t) - D^s S_r^n x(t)| = o(1)$ for $s = 0, 1, \ldots, r-1$, uniformly in t on a sequence of imbedded meshes. In [8, Theorems 6, 8 and 10] the cases $p = r$ and $p = 2r$ of Theorem 9.1 (for more general meshes and more general types of splines) are proved.

10. A REPRODUCING KERNEL

As remarked before, the space $\mathscr{W}_r$ plays a particular role in the analysis of $2r$-splines. By $\mathring{\mathscr{W}} = \mathring{\mathscr{W}}_r^n$ we denote the subspace of $\mathscr{W}_r$ whose elements x satisfy the conditions

$$x(\nu/n) = 0, \qquad \nu = 0, 1, \ldots, n-1. \qquad (10.1)$$

$\mathring{\mathscr{W}}$ is a Hilbert space which has a reproducing kernel, that is, a function $K_\tau \in \mathring{\mathscr{W}}$ such that

$$x(\tau) = (x, K_\tau)_{\mathscr{W}_r} = \int_0^1 x^{(r)}(t)\,\overline{K_\tau^{(r)}(t)}\,dt \tag{10.2}$$

for each $x \in \mathring{\mathscr{W}}$ and each real τ. In this section we find explicit expressions for K_τ.

r-fold integration by parts in (10.2) shows that K_τ is the reproducing kernel of $\mathring{\mathscr{W}}$ if it satisfies, for $\tau \neq 0, \pm 1/n, \pm 2/n, \ldots$, the following conditions

$$\begin{aligned} &\text{(i)}\ K_\tau \in \mathscr{C}_{2r-2},\ K_\tau(t+1) = K_\tau(t), \quad -\infty < t < \infty \\ &\text{(ii)}\ K_\tau(\nu/n) = 0, \quad \nu = 0, 1, \ldots, n-1 \\ &\text{(iii)}\ K_\tau^{(2r)}(t) = 0, \quad t \neq 0,\ \pm 1/n,\ \pm 2/n, \ldots;\ t \neq \tau \\ &\text{(iv)}\ K_\tau^{(2r-1)}(\tau+0) - K_\tau^{(2r-1)}(\tau-0) = (-1)^r, \quad -\infty < \tau < \infty. \end{aligned} \tag{10.3}$$

The function

$$C_\tau(t) = [(-1)^r/(2r)!]\,[\mathring{B}_{2r}(t) - \mathring{B}_{2r}(t-\tau)] \tag{10.4}$$

is seen, by the use of (2.5), to satisfy (10.3, (iii), (iv)). To obtain a function that also satisfies (10.3(ii)) we subtract the spline interpolant $S_r^n C_\tau(t)$, obtaining

$$K_\tau(t) = C_\tau(t) - \sum_{\nu=0}^{n-1} C_\tau(\nu/n)\, s_0(t - \nu/n). \tag{10.5}$$

Clearly, K_τ satisfies (10.3); hence is the reproducing kernel of $\mathring{\mathscr{W}}$. We develop a more explicit expression for K_τ.

By definition of s_0 [see (2.25)] and by the use of (2.6), we have

$$\begin{aligned} &\sum_{\nu=0}^{n-1} \mathring{B}_{2r}(\tau - \nu/n)\, s_0(t - \nu/n) \\ &= n^{-1} \sum_{\nu=0}^{n-1} \mathring{B}_{2r}(\tau - \nu/n) + \sum_{\mu,\nu=0}^{n-1} (\rho_\mu - n^{2r-2}/B_{2r})\, \mathring{B}_{2r}(\tau - \nu/n)\, \mathring{B}_{2r}(t + \overline{\mu - \nu}/n) \\ &= n^{-2r} \mathring{B}_{2r}(n\tau) - n^{-2r} \mathring{B}_{2r}(nt)\, \mathring{B}_{2r}(n\tau)/B_{2r} + \sum_{\mu,\nu=0}^{n-1} \rho_{\mu-\nu}\, \mathring{B}_{2r}(t - \mu/n)\, \mathring{B}_{2r}(\tau - \nu/n). \end{aligned} \tag{10.6a}$$

Hence, using $\sum_\nu \rho_{\mu-\nu} B_{2r}(\nu/n) = \delta_{0,\mu}$, which is a result from (2.25):

$$\begin{aligned} &\sum_{\nu=0}^{n-1} [\mathring{B}_{2r}(\tau - \nu/n) - \mathring{B}_{2r}(\nu/n)]\, s_0(t - \nu/n) \\ &= n^{-2r}[\mathring{B}_{2r}(nt) + \mathring{B}_{2r}(n\tau) - \mathring{B}_{2r}(nt)\, \mathring{B}_{2r}(n\tau)/B_{2r} - B_{2r}] \\ &\qquad - B_{2r}(t) + \sum_{\mu,\nu=1}^{n-1} \rho_{\mu-\nu}\, \mathring{B}_{2r}(t - \mu/n)\, \mathring{B}_{2r}(\tau - \nu/n). \end{aligned} \tag{10.6b}$$

Therefore, (10.5) gives

$$K_\tau(t) = [(-1)^r/(2r)!]\,[n^{-2r}(\mathring{B}_{2r}(nt) + \mathring{B}_{2r}(n\tau) - \mathring{B}_{2r}(nt)\,\mathring{B}_{2r}(n\tau)/B_{2r} - B_{2r}) + \sum_{\mu,\nu=0}^{n-1} \rho_{\mu-\nu}\,\mathring{B}_{2r}(t-\mu/n)\,\mathring{B}_{2r}(\tau-\nu/n) - \mathring{B}_{2r}(t-\tau)]. \quad (10.7)$$

This formula makes the symmetry of the kernel apparent.

We develop still another formula for K_τ, using the functions b_ν for this purpose. By (2.20), (2.6) and (5.1), we have

$$\begin{aligned}\sum_{\mu,\nu=0}^{n-1} \rho_{\mu-\nu}\,\mathring{B}_{2r}(t-\mu/n)\,\mathring{B}_{2r}(\tau-\nu/n) &= n^{-1}\sum_{m=0}^{n-1}\sum_{\mu,\nu=0}^{n-1} \lambda_m^{-1}\,\epsilon_n^{m(\mu-\nu)}\,\mathring{B}_{2r}(t-\mu/n)\,\mathring{B}_{2r}(\tau-\nu/n) \quad (10.8)\\ &= n^{-1}\sum_{m=0}^{n-1} \lambda_m\, b_m(t)\,\overline{b_m(\tau)} + n^{-2r}\,\mathring{B}_{2r}(nt)\,\mathring{B}_{2r}(n\tau)/B_{2r}.\end{aligned}$$

If this is used in (10.7), we obtain

$$K_\tau(t) = [(-1)^r/(2r)!]\,[n^{-2r}(\mathring{B}_{2r}(nt) + \mathring{B}_{2r}(n\tau) - B_{2r}) + n^{-1}\sum_{\nu=0}^{n-1} \lambda_\nu\, b_\nu(t)\,\overline{b_\nu(\tau)} - \mathring{B}_{2r}(t-\tau)]. \quad (10.9)$$

We also give the Fourier expansion of K_τ. Using (2.7) and (5.8) in (10.9), one arrives at

$$K_\tau(t) = [(-1)^r/(2r)!]\,n^{-2r}(\mathring{B}_{2r}(n\tau) - B_{2r}) + (2\pi)^{-2r} \sum_k{}' k^{-2r}(e^{-2\pi ik\tau} - \overline{b_k(\tau)})\,e^{2\pi ikt}. \quad (10.10)$$

11. Exact Error Bounds

Let $u(x)$ be a linear functional defined for a class of functions x that includes $\mathring{\mathscr{W}}_r^n$ (for definition see Section 10), and which is bounded on $\mathring{\mathscr{W}}_r^n$. Let its bound be denoted by $\|u\| = \|u\|_{\mathring{\mathscr{W}}_r^n}$; thus:

$$\|u\|_{\mathring{\mathscr{W}}_r^n} = \sup_{\substack{x\in\mathring{\mathscr{W}}_r^n\\ \|x\|_{\mathscr{W}_r}\leq 1}} |u(x)|. \quad (11.1)$$

Using the reproducing kernel K_τ of $\mathring{\mathscr{W}}_r^n$ (see Section 10), we have $\overline{u(\bar{x})} = (x, u(K))$; hence

$$\|u\| = \|u(K)\|_{\mathscr{W}_r} = (u(\overline{u(K)}))^{1/2}. \quad (11.2)$$

It follows from general theory (see [1] or [2]) that $u(Sx)$, where $Sx = S_r^n x$ is the spline interpolant of x, represents the median of the values of $u(x)$ for x in the class

$$\mathscr{D} = \mathscr{D}_r^n(\xi;\rho) : \|x\|_{\mathscr{W}_r}^2 \leq \rho^2,\ x(\nu/n) = \xi_\nu, \qquad \nu = 0, 1, \ldots, n-1. \quad (11.3)$$

($\mathscr{D}$ is a "disk" in $\mathscr{W}_r$), and that the maximal deviation of the values $u(x)$ from the median in $\mathscr{D}$ is

$$\sup_{x\in\mathscr{D}} |u(x) - u(Sx)| = \|u\|(\rho^2 - \|Sx\|^2_{\mathscr{W}_r})^{1/2}. \tag{11.4}$$

We shall calculate $\|Sx\|_{\mathscr{W}_r}$, and $\|u\|_{\mathscr{W}_r,n}$ for various functionals u.

a. By (5.6) and (5.8)

$$\|Sx\|^2_{\mathscr{W}_r} = (-1)^{r-1}(2r)!\,n \sum_{\nu=0}^{n-1} \lambda_\nu^{-1}|\hat{\xi}_\nu|^2,\ \hat{\xi}_\nu = n^{-1}\sum_{\mu=0}^{n-1} \epsilon_n^{-\mu\nu}\xi_\mu. \tag{11.5a}$$

This is an explicit expression for $\|Sx\|_{\mathscr{W}_r}$. We may also use the spline approximation of the Fourier coefficients to express $\|Sx\|_{\mathscr{W}_r}$. We denote them by $\hat{\xi}_{\nu,r}$, and have, by (4.7), for $\nu \not\equiv 0 \pmod n$

$$\hat{\xi}_{\nu,r} = \int_0^1 Sx(t)\,e^{-2\pi i\nu t}\,dt = (-1)^{r-1}(2r)!\,(2\pi\nu)^{-2r}\lambda_\nu^{-1}n\hat{\xi}_\nu. \tag{11.6}$$

Therefore, (11.5) becomes

$$\|Sx\|^2_{\mathscr{W}_r} = \sum_{\nu=0}^{n-1}(2\pi\nu)^{2r}\,\hat{\xi}_{\nu,r}\,\bar{\hat{\xi}}_\nu \qquad r = 1, 2, \ldots. \tag{11.5b}$$

b. Let $u(x) = u_\tau(x) = x(\tau)$; that is, we consider interpolation at the point τ, and $\|u_\tau\|$ is a significant measure for the error in interpolation. Since $u_\tau(K) = K_\tau$, (11.2) gives

$$\|u_\tau\| = \|K_\tau\|_{\mathscr{W}_r}.$$

To calculate this we use (10.9), according to which

$$(-1)^r K_\tau^{(r)}(t) = (n^{-r}/r!)\,\mathring{B}_r(nt) - (1/r!)\,\mathring{B}_r(t-\tau) + (n^{-1}/(2r)!)\sum_{\nu=0}^{n-1}\lambda_\nu b_\nu^{(r)}(t)\,\overline{b_\nu(\tau)}. \tag{11.7}$$

If this function is expanded in a Fourier series, the coefficient of $\exp(2\pi i\nu t)$ is 0 if $\nu = 0$, and is found to be

$$(-1)^{r/2}(2\pi\nu)^{-r}(e^{-2\pi i\nu\tau} - \overline{b_\nu(\tau)}),$$

by (2.7) and (5.10), if $\nu = \pm 1, \pm 2, \ldots$. Therefore

$$\|u_\tau\| = (2\pi)^{-r}\left\{\sum_\nu{}' \nu^{-2r}|e^{2\pi i\nu\tau} - b_\nu(\tau)|^2\right\}^{1/2}. \tag{11.8}$$

Although in deriving this we assumed r to be even, it also holds for r odd. Clearly, $|\exp(2\pi i\nu\tau) - b_\nu(\tau)| = |\exp(-2\pi i\nu\tau) - b_{-\nu}(\tau)|$; hence (11.8) may also be written as

$$\|u_\tau\| = (2\pi)^{-r}\left\{2\sum_{\nu=1}^{\infty}\nu^{-2r}|e^{2\pi i\nu\tau} - b_\nu(\tau)|^2\right\}^{1/2}. \tag{11.9}$$

We evaluate the order of $\|u_\tau\|_{\mathscr{W}_r{}^n}$ as $n \to \infty$. By (5.29), we obtain from (11.9):

$$\tfrac{1}{2}(2\pi)^{2r}\|u_\tau\|^2 = \sum_{1 \leqslant \nu \leqslant [n/2]} \nu^{-2r}|e^{2\pi i\nu\tau} - b_\nu(\tau)|^2 + \sum_{\nu > [n/2]} \nu^{-2r}|e^{2\pi i\nu\tau} - b_\nu(\tau)|^2$$
$$\leqslant 2^{4r+2}n^{-4r} \sum_{1 \leqslant \nu \leqslant [n/2]} \nu^{-2r} + 4 \sum_{\nu > [n/2]} \nu^{-2r}$$
$$\leqslant 2^{2r+1}n^{-2r+1} + 4(n/2)^{-2r+1}/(2r-1). \tag{11.10}$$

Thus,

$$\|u_\tau\|_{\mathscr{W}_r{}^n} \leqslant 2^{3/2}\pi^{-r}n^{-r+1/2} \qquad r = 1, 2, \ldots; n = 1, 2, \ldots. \tag{11.11}$$

The result $\|u_\tau\| = 0(n^{-r+1/2})$ was proved by Weinberger [9] for the case $r = 2$, x nonperiodic.

We now show that $0(n^{-r+1/2})$ is the exact order of $\sup_\tau \|u_\tau\|_{\mathscr{W}_r{}^n}$. Using $\tau = 1/2n$ and $\nu = \kappa n$ $(0 < \kappa < 1)$, we have by (5.8)

$$n^{r-1/2}\nu^{-r}|e^{\pi i\nu/n} - b_\nu(1/2n)| = C(\kappa)n^{-1/2} \tag{11.12}$$

where we have set

$$C(\kappa) = 2\kappa^{-r}\sum_{k \text{ odd}}(k-\kappa)^{-2r}\Big/\sum_k (k-\kappa)^{-2r}. \tag{11.13}$$

Let $C_0 > 0$ be chosen such that

$$C(\kappa) \geqslant C_0, \quad \tfrac{1}{2} \leqslant \kappa \leqslant \tfrac{3}{4}. \tag{11.14}$$

Then by (11.12)

$$n^{2r-1}\sum_{\nu=[n/2]|}^{[3n/4]} \nu^{-2r}|e^{\pi i\nu/n} - b_\nu(1/2n)|^2 \geqslant \tfrac{1}{4}C_0{}^2 \tag{11.15}$$

and by (11.9)

$$n^{r-1/2}\|u_{1/2n}\|_{\mathscr{W}_r{}^n} > C_0(2\pi)^{-r}2^{-1/2}, \quad n = 1, 2, \ldots; r = 1, 2, \ldots, \tag{11.16}$$

which proves the assertion. By using the inequalities

$$\sum_{k \text{ odd}}(k-\kappa)^{-2r}\Big/\sum_k(k-\kappa)^{-2r} > \sum_{k \text{ odd}}(k-\kappa)^{-2r}\Big/[\kappa^{-2r} + 2\sum_{k \text{ odd}}(k-\kappa)^{-2r}]$$
$$> (1-\kappa)^{-2r}/[\kappa^{-2r} + 2(1-\kappa)^{-2r}] \geqslant 1/3, \quad \tfrac{1}{2} \leqslant \kappa \leqslant \tfrac{3}{4}$$

we see that $C_0 = 2^{2r+1}3^{-r-1}$ satisfies (11.14). Thus (11.16) becomes

$$n^{r-1/2}\|u_{1/2n}\|_{\mathscr{W}_r{}^n} > (2/3)^{r+1}\pi^{-r}2^{-1/2}, \quad n = 1, 2, \ldots; r = 1, 2, \ldots. \tag{11.17}$$

We now prove the existence and determine the value of

$$\lim_{n\to\infty} n^{r-1/2}\|u_{1/2n}\|_{\mathscr{W}_r{}^n}. \tag{11.18}$$

By (5.8)

$$|e^{\pi i\nu/n} - b_\nu(1/2n)| = 2\left|\sum_{k \text{ odd}}(k-\nu/n)^{-2r}\Big/\sum_k(k-\nu/n)^{-2r}\right|, \quad \nu \not\equiv 0 \pmod n$$
$$= 2, \quad \nu = n, 3n, 5n, \ldots,$$
$$= 0, \quad \nu = 0, 2n, 4n, \ldots. \tag{11.19}$$

We introduce the functions

$$C_s(z) = z^{-s} + \sum_k{}' (z-k)^{-s}, \quad s = 1, 2, \ldots. \tag{11.20}$$

Then

$$C_s(\tfrac{1}{2}z) = 2^s \sum_{k \text{ even}} (z-k)^{-s}, \quad C_s(\tfrac{1}{2}z + \tfrac{1}{2}) = 2^s \sum_{k \text{ odd}} (z-k)^{-s}. \tag{11.21}$$

Substitution of (11.19), (11.20) and (11.21) in (11.8) yields

$$\|u_{1/2n}\| = (2\pi)^{-r} \left\{ 2^{-4r+2} \sum_{\nu \neq 0} \nu^{-2r} C_{2r}^2(\nu/2n + \tfrac{1}{2})/C_{2r}^2(\nu/n) + 2^{-2r+2} n^{-2r} C_{2r}(\tfrac{1}{2}) \right\}^{1/2}. \tag{11.22}$$

Since $C_s(z+1) = C_s(z)$, one finds

$$\begin{aligned}
&\sum_{\nu \neq 0} \nu^{-2r} C_{2r}^2(\nu/2n + \tfrac{1}{2})/C_{2r}^2(\nu/n) \\
&= \sum_{\nu=1}^{n-1} \left[C_{2r}^2(\nu/2n + \tfrac{1}{2}) \sum_{k \text{ even}} (\nu + kn)^{-2r} + C_{2r}^2(\nu/2n) \sum_{k \text{ odd}} (\nu + kn)^{-2r} \right] \Big/ C_{2r}^2(\nu/n) \\
&= n^{-2r} 2^{-2r} \sum_{\nu=1}^{n-1} \left[C_{2r}^2(\nu/2n + \tfrac{1}{2})\, C_{2r}(\nu/2n) + C_{2r}^2(\nu/2n)\, C_{2r}(\nu/2n + \tfrac{1}{2}) \right] \Big/ C_{2r}^2(\nu/n) \\
&= n^{-2r} \sum_{\nu=1}^{n-1} C_{2r}(\nu/2n)\, C_{2r}(\nu/2n + \tfrac{1}{2})/C_{2r}(\nu/n).
\end{aligned} \tag{11.23}$$

Therefore, (11.22) may be written as

$$n^{r-1/2}\|u_{1/2n}\| = (2\pi)^{-r} 2^{-2r+1} \left\{ \frac{1}{n} \sum_{\nu=1}^{n-1} C_{2r}(\nu/2n)\, C_{2r}(\nu/2n + \tfrac{1}{2})/C_{2r}(\nu/n) + 2^{2r} n^{-1} C_{2r}(\tfrac{1}{2}) \right\}^{1/2}. \tag{11.24}$$

$C_s(z)$ is a meromorphic function with poles of order s at $z = 0, \pm 1, \pm 2, \ldots$. From the well known Mittag–Leffler expansion of cotangent, one obtains

$$C_s(z) = [(-1)^{s-1} \pi^s/(s-1)!] \cot^{(s-1)} \pi z. \tag{11.25}$$

The function $C_{2r}(\tfrac{1}{2}t)\, C_{2r}(\tfrac{1}{2}t + \tfrac{1}{2})/C_{2r}(t)$ occurring in (11.24) is analytic in $0 \leqslant t \leqslant 1$. Indeed, it approaches the value $2^{4r} \sum_{k \text{ odd}} k^{-2r}$ both as t approaches 0 or 1. Therefore, (11.24) is the Riemann sum of a convergent integral, and one obtains

$$\lim_{n \to \infty} n^{r-1/2}\|u_{1/2n}\|_{\mathscr{W}_r{}^n} = (2\pi)^{-r} 2^{-2r+1} \left\{ \int_0^1 dt\, C_{2r}(\tfrac{1}{2}t + \tfrac{1}{2})\, C_{2r}(\tfrac{1}{2}t)/C_{2r}(t) \right\}^{1/2}. \tag{11.26}$$

We have proved

THEOREM 11.1. $2^{-1/2}(2/3)^{r+1} < \pi^r n^{r-1/2} \sup |x(\tau)| < 2^{3/2}$, $n = 1, 2, \ldots$; $r = 1, 2, \ldots$ *if the supremum is taken over* $-\infty < \tau < \infty$ *and over the class of functions*

x of period 1 which vanish at 0, $\pm 1/n$, $\pm 2/n$, ... and for which $\int_0^1 |x^{(r)}(t)|^2 dt < 1$. *Moreover,* $n^{r-1/2} \sup |x(1/2n)|$ *approaches a positive limit as* $n \to \infty$, *given in* (11.26).

c. We now assume $r \geqslant 2$ and consider $v(x) = v_\tau(x) = x'(\tau)$. Then $v(K) = (d/d\tau) K_\tau$, and (11.2) gives

$$\|v_\tau\| = \|dK_\tau/d\tau\|_{\mathscr{W}_r}. \tag{11.27}$$

By (11.7) the coefficient of $\exp(2\pi i\nu t)$ in the expansion of $(d/d\tau) K_\tau(t)$ is found to be

$$(-1)^{r/2}(2\pi\nu)^{-r}(-2\pi i\nu e^{-2\pi i\nu\tau} - \overline{b_\nu'(\tau)}).$$

Thus,

$$\|v_\tau\| = (2\pi)^{-r+1}\left\{2\sum_{\nu=1}^{\infty} \nu^{-2r+2}|e^{2\pi i\nu\tau} - b_\nu'(\tau)/2\pi i\nu|^2\right\}^{1/2}. \tag{11.28}$$

One proceeds as above to show that $\|v_\tau\|_{\mathscr{W}_r^n} = O(n^{-r+3/2})$ uniformly in τ. In the same way one can prove that if $r \geqslant s+1$, and $v_\tau(x)$ represents $x^{(s)}(\tau)$, then $\|v_\tau\|_{\mathscr{W}_r^n} = O(n^{-r+s+1/2})$ uniformly in τ, and that this is the exact asymptotic order.

For the case where $v_0(x)$ represents $x'(0)$, we prove the existence and determine the value of

$$\lim_{n\to\infty} n^{r-3/2}\|v_o\|_{\mathscr{W}_r^n}. \tag{11.29}$$

By (5.8),

$$\begin{aligned} 1 - b_\nu'(0)/2\pi i\nu &= (n/\nu)\sum_{k=-\infty}^{\infty} k(k-\nu/n)^{-2r}\Big/\sum_{k=-\infty}^{\infty}(k-\nu/n)^{-2r} \\ &= [C_{2r}(\nu/n) - (n/\nu)\, C_{2r-1}(\nu/n)]/C_{2r}(\nu/n), \quad \nu \not\equiv 0 \pmod n \\ &= 1, \quad \nu \equiv 0 \pmod n \end{aligned} \tag{11.30}$$

where we have used the functions (11.20). Substitution of (11.30) in (11.28) yields

$$\begin{aligned} \|v_o\| = (2\pi)^{-r-1}\Big\{&\sum_\nu{}' \,[n^2\nu^{-2r} C_{2r-1}^2(\nu/n) - 2n\nu^{-2r+1} C_{2r-1}(\nu/n)\, C_{2r}(\nu/n) \\ &+ \nu^{-2r+2} C_{2r}^2(\nu/n)]/C_{2r}^2(\nu/n) + n^{-2r+2}\sum_k{}' k^{-2r+2}\Big\}^{1/2}, \end{aligned} \tag{11.31}$$

and since $C_s(z+1) = C_s(z)$,

$$\begin{aligned} n^{r-3/2}\|v_o\| = (2\pi)^{-r-1}\Big\{&n^{-1}\sum_{\nu=1}^{n-1}[C_{2r-1}^2(\nu/n) - 2C_{2r-1}^2(\nu/n) \\ &+ C_{2r-2}(\nu/n)\, C_{2r}(\nu/n)]/C_{2r}(\nu/n) + n^{-2r+2}\sum_k{}' k^{-2r+2}\Big\}^{1/2}. \end{aligned} \tag{11.32}$$

The function $[-C_{2r-1}^2(t)+C_{2r-2}(t)\,C_{2r}(t)]/C_{2r}(t)$, occurring in (11.32), is analytic in $0 \leqslant t \leqslant 1$. It approaches the value $\sum_k' k^{-2r+2}$ as t approaches 0. Therefore, (11.32) is the Riemann sum of a convergent integral, and one obtains

$$\lim_{n\to\infty} n^{r-3/2}\|v_0\|_{\mathscr{W}_r,n} = (2\pi)^{-r-1}\left\{\int_0^1 dt\,[C_{2r-2}(t)\,C_{2r}(t) - C_{2r-1}^2(t)]/C_{2r}(t)\right\}^{1/2}$$

$$r = 2, 3, \ldots. \qquad (11.33)$$

We have proved

THEOREM 11.2. *There are positive numbers c_r, C_r depending on r only, such that for $s = 0, 1, \ldots, r-1$*

$$c_r < n^{r-s-1/2} \sup |x^{(s)}(\tau)| < C_r, \qquad n = 1, 2, \ldots;\ r = 2, 3, \ldots$$

if the supremum is taken over $-\infty < \tau < \infty$ and over the class of functions x of period 1 *which vanish at* 0, $\pm 1/n$, $\pm 2/n$, ... *and for which $\int_0^1 |x^{(r)}(t)|^2\,dt \leqslant 1$. Moreover, $n^{r-3/2}\sup|x'(0)|$ approaches a positive limit as $n \to \infty$, given in* (11.33).

d. For the quadrature functional $w(x) = w_\tau(x) = \int_{-\tau}^{\tau} x(t)\,dt$, we have $w(K) = \int_{-\tau}^{\tau} K_\sigma\,d\sigma$, and (11.2) gives

$$\|w_\tau\| = \left\|\int_{-\tau}^{\tau} K_\sigma\,d\sigma\right\|_{\mathscr{W}_r}. \qquad (11.34)$$

Using (11.7), this gives

$$\|w_\tau\| = 2(2\pi)^{-r-1}\left\{2\sum_{\nu=1}^{\infty} \nu^{-2r-2}(\sin 2\pi\nu\tau - \pi\nu \int_{-\tau}^{\tau} b_\nu(t)\,dt)^2\right\}^{1/2}. \quad (11.35)$$

We work out the order of $\|w_\tau\|_{\mathscr{W}_r,n}$ as $n \to \infty$ for the case $\tau = 1/n$. By (6.22) we have

$$\begin{aligned} &\sin(2\pi\nu/n) - \pi\nu\int_{-1/n}^{1/n} b_\nu(t)\,dt \\ &\quad = \sin(2\pi\nu/n)\sum_k{}' k(k-\nu/n)^{-2r-1}\Big/\sum(k-\nu/n)^{-2r}, \quad \nu \not\equiv 0 \,(\mathrm{mod}\, n), \\ &\quad = -2\pi\nu/n, \quad \nu \equiv 0\,(\mathrm{mod}\, n). \end{aligned} \qquad (11.36)$$

Therefore,

$$\|w_{1/n}\| = 2(2\pi)^{-r-1}\left\{2\sum_{\nu\geqslant 1,\,\nu\not\equiv 0} \nu^{-2r-2}\sin^2(2\pi\nu/n)\left[\sum_k k(k-\nu/n)^{-2r-1}\Big/\sum_k (k-\nu/n)^{-2r}\right]^2 + 8\pi^2 n^{-2r-2}\sum_{\nu\geqslant 1}\nu^{-2r}\right\}^{1/2}. \qquad (11.37)$$

We use $\sin^2(2\pi\nu/n) < 4\pi^2\nu^2/n^2$ in (11.37), and for $2\nu \leqslant n$ the inequalities

$$0 \leqslant \sum_k k(k-\nu/n)^{-2r-1}\Big/\sum_k (k-\nu/n)^{-2r} \leqslant (\nu/n)^{2r}\sum_k k(k-\nu/n)^{-2r-1}$$

$$\leqslant (\nu/n)^{2r}\left[\sum_{k=1}^{\infty} k^{-2r} + \sum_{k=1}^{\infty} k(k-\tfrac{1}{2})^{-2r-1}\right]$$

$$= (\nu/n)^{2r}\left[2^{2r}\sum_{k=1}^{\infty} k^{-2r} + (2^{2r}-\tfrac{1}{2})\sum_{k=1}^{\infty} k^{-2r-1}\right]$$

$$< 2^{2r+1}(\nu/n)^{2r}. \tag{11.38}$$

For $2\nu > n$, we have, more directly, by (5.12)

$$\left|\sin(2\pi\nu/n) - \pi\nu\int_{-1/n}^{1/n} b_\nu(t)\,dt\right| \leqslant 4\pi\nu/n. \tag{11.39}$$

Thus,

$$\|w_{1/n}\| < 2(2\pi)^{-r-1}\left\{2^{4r+5}\pi^2 n^{-4r-2}\sum_{2\nu\leqslant n}\nu^{2r} + 2^5\pi^2 n^{-2}\sum_{2\nu>n}\nu^{2r} + 8\pi^2 n^{-2r-2}\sum_{\nu\geqslant 1}\nu^{-2r}\right\},$$

or

$$\|w_{1/n}\|_{\mathscr{W}_r^n} < 2^{7/2}\pi^{-r}n^{-r-1/2}, \quad n = 1,2,\ldots;\ r = 1,2,\ldots. \tag{11.40}$$

To show that $n^{r+1/2}\|w_{1/n}\|_{\mathscr{W}_r^n}$ tends to a positive limit, we make use of the functions (11.20) and write

$$\sum_{\nu=0}\nu^{-2r-2}\sin^2(2\pi\nu/n)\left[\sum_k k(k-\nu/n)^{-2r-1}\Big/\sum_k (k-\nu/n)^{-2r}\right]^2$$

$$= n^{-2r-2}\sum_{\nu=1}^{n-1}\sin^2(2\pi\nu/n)\,[-C_{2r+1}^2(\nu/n) + C_{2r}(\nu/n)\,C_{2r+2}(\nu/n)]/C_{2r}(\nu/n). \tag{11.41}$$

Substitution of (11.41) in (11.37) yields

$$n^{r+1/2}\|w_{1/n}\| = 2(2\pi)^{-r-1}\left\{n^{-1}\sum_{\nu=1}^{n-1}\sin^2(2\pi\nu/n)\,[-C_{2r+1}^2(\nu/n) + C_{2r}(\nu/n)\,C_{2r+2}(\nu/n)]/C_{2r}(\nu/n) + 8\pi^2 n^{-1}\sum_{\nu=1}^{\infty}\nu^{-2r}\right\}^{1/2}. \tag{11.42}$$

From this one concludes as above

$$\lim_{n\to\infty} n^{r+1/2}\|w_{1/n}\|_{\mathscr{W}_r^n} = 2(2\pi)^{-r-1}\left\{\int_0^1 dt\,\sin^2(2\pi t)\,[C_{2r}(t)\,C_{2r-2}(t) - C_{2r+1}^2(t)]/C_{2r}(t)\right\}^{1/2}. \tag{11.43}$$

The integrand is analytic in $[0,1]$. It approaches the value $4\pi^2\sum' k^{-2r}$ as t approaches 0. Thus, the limit in (11.43) is not 0. We have proved

THEOREM 11.3. $\sup\left|\int_{-1/n}^{1/n} x(t)\,dt\right| < 2^{7/2}\pi^{-r}n^{-r-1/2}$, $n = 1, 2, \ldots;\ r = 1, 2, \ldots$ *if the supremum is taken over the class of functions x of period 1 which vanish at 0, $\pm 1/n$, $\pm 2/n$, ... and for which $\int_0^1 |x^{(r)}(t)|^2\,dt \leq 1$. Moreover, $n^{r+1/2}\sup\left|\int_{-1/n}^{1/n} x(t)\,dt\right|$ approaches a positive limit as $n \to \infty$, given in* (11.43).

e. Finally we consider the Fourier coefficient functional

$$f_\nu(x) = \int_0^1 x(t)\,e^{-2\pi i\nu t}\,dt, \quad \nu = 0, \pm 1, \pm 2, \ldots. \tag{11.44}$$

By (10.10) we have

$$\begin{aligned} \overline{f_\nu(K)} &= (2\pi\nu)^{-2r}[e^{2\pi i\nu t} - b_\nu(t)], & \nu \neq 0 \\ &= [(-1)^r/(2r)!]\,n^{-2r}[\mathring{B}_{2r}(nt) - B_{2r}], & \nu = 0. \end{aligned} \tag{11.45}$$

Using this in (11.2), we find

$$\begin{aligned} \|f_\nu\|_{\mathscr{W}_{r,n}} &= (2\pi n)^{-r}\left\{\sum_k{}' (k - \nu/n)^{-2r}/[1 + (\nu/n)^{2r}\sum_k{}' (k - \nu/n)^{-2r}]\right\}^{1/2}, \\ &\qquad\qquad \nu \not\equiv 0 \,(\operatorname{mod} n) \\ &= (2\pi\nu)^{-r}, \quad \nu \neq 0,\ \nu \equiv 0\,(\operatorname{mod} n) \\ &= (2\pi n)^{-r}\left\{\sum_k{}' k^{-2r}\right\}^{1/2} = n^{-r}\{|B_{2r}|/(2r)!\}^{1/2}, \quad \nu = 0. \end{aligned} \tag{11.46}$$

Clearly, $\|f_\nu\|$ is of order $O(n^{-r})$. More precisely,

$$\lim_{n\to\infty} n^r\|f_\nu\|_{\mathscr{W}_{r,n}} = \{|B_{2r}|/(2r)!\}^{1/2}, \quad \nu = 0, \pm 1, \pm 2, \ldots. \tag{11.47}$$

It is noteworthy that this limit is independent of ν. We have proved

THEOREM 11.4

$$\lim_{n\to\infty} n^r \sup\left|\int_0^1 x(t)\,e^{-2\pi i\nu t}\,dt\right| = \{|B_{2r}|/(2r)!\}^{1/2}, \quad r = 1, 2, \ldots,$$

if the supremum is taken over the class of functions x of period 1 which vanish at 0, $\pm 1/n$, $\pm 2/n$, ... and for which $\int_0^1 |x^{(r)}(t)|^2\,dt \leq 1$.

We shall now show that $x_\nu(t) = \exp(2\pi i\nu t)$ is an extremal function for the approximation of the value $f_\nu(x)$, given $x \in \mathscr{D}$. That is, equality holds in (11.4) for $x = x_\nu$ if $u = f_\nu$ and $\rho^2 = \|x_\nu\|^2_{\mathscr{W}_r} = (2\pi\nu)^{2r}$. Indeed, since $Sx_\nu = b_\nu$, we have by (5.11)

$$(\rho^2 - \|Sx_\nu\|^2_{\mathscr{W}_r})^{1/2} = \left\{(2\pi\nu)^{2r} - (2\pi n)^{2r}\Big/\sum_k (k - \nu/n)^{-2r}\right\}^{1/2}, \quad \nu \not\equiv 0\,(\operatorname{mod} n)$$

and by (11.46)

$$\|f_\nu\|(\rho^2 - \|Sx_\nu\|^2_{\mathscr{W}_r})^{1/2} = 1 - (\nu/n)^{-2r} \Big/ \sum_k (k - \nu/n)^{-2r}, \quad \nu \not\equiv 0 \,(\mathrm{mod}\, n). \tag{11.48}$$

On the other hand, by (5.8)

$$f_\nu(x_\nu) - f_\nu(Sx_\nu) = 1 - (\nu/n)^{-2r} \Big/ \sum_k (k - \nu/n)^{-2r}, \quad \nu \not\equiv 0 \,(\mathrm{mod}\, n). \tag{11.49}$$

Thus, we have proved, for $\nu \not\equiv 0$ (mod n)

$$f_\nu(x_\nu) - f_\nu(S_r^n x_\nu) = \|f_\nu\|_{\mathscr{W}_r^n}\{\|x_\nu\|^2_{\mathscr{W}_r} - \|S_r^n x_\nu\|^2_{\mathscr{W}_r}\}^{1/2}. \tag{11.50}$$

For $\nu = 0$, both sides of (11.50) are equal to 0, and for $\nu = kn$ ($k = \pm 1, \pm 2, \ldots$), both sides are equal to 1. Thus (11.50) is valid for every ν. In summary, we have

THEOREM 11.5. *Let $\mathscr{D}_r^n$ ($n = 1, 2, \ldots$, $r = 1, 2, \ldots$) be the class of functions of period* 1 *which have fixed* (*real or complex*) *values at* 0, $\pm 1/n$, $\pm 2/n$, ... *and for which* $\int_0^1 |x^{(r)}(t)|^2 \, dt \leqslant 1$. *Then the median value of the Fourier coefficient* $\int_0^1 x(t) e^{-2\pi i \nu t} \, dt$ *is* 0 *if* $\nu = kn$ ($k = \pm 1, \pm 2, \ldots$); *otherwise, it is*

$$\hat{\xi}_{\nu,r}(x) = (1/n) \sum_{m=0}^{n-1} x(m/n) \, e^{-2\pi i \nu m/n} \Big/ \sum_k (1 - kn/\nu)^{-2r}.$$

The least upper bound of the deviation of the median from the true value in $\mathscr{D}_r^n$ *is*

$$\|f_\nu\| \left\{ 1 - \sum_{\nu=1}^{n-1} (2\pi\nu)^{2r} \, \hat{\xi}_{\nu,r}(x) \, \overline{\hat{\xi}_\nu(x)} \right\}^{1/2}$$

where

$$\hat{\xi}_\nu(x) = (1/n) \sum_{m=0}^{n-1} x(m/n) \, e^{-2\pi i \nu m/n}$$

and $\|f_\nu\|$ *is given in* (11.46). *The coefficient* $\|f_\nu\|$ *tends to* 0 *like* $O(n^{-r})$ *as* $n \to \infty$, *and*

$$\lim_{n \to \infty} n^r \|f_\nu\| = \{|B_{2r}|/(2r)!\}^{1/2},$$

independent of ν. *The least upper bound is attained by* $x(t) = (2\pi\nu)^{-r} \exp(2\pi i \nu t)$ *in* $\mathscr{D}_r^n$.

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